Ex-Post Constrained-Efficient Bilateral Trade with Risk-Averse Traders

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Abstract

We address robust mechanism design for bilateral trade of an indivisible commodity, under incomplete information on traders' private reservation values. Under *ex-post* individual rationality and *ex-post* incentive compatibility, we define the notion of *ex-post* constrained-efficiency. It is weaker than *interim* constrained efficiency, and it is a notion of constrained optimality that is independent of the details of the distribution of types. When traders are risk neutral the class of *ex-post* constrained-efficient mechanisms is equivalent to probability distributions over posted prices. In general, among mechanisms satisfying incentive and participation constraints, a sufficient condition for constrained efficiency is simplicity: for each draw of types the outcome is a lottery between trade at one type-contingent price and no trade. For environments with constant relative risk aversion, we characterize simple mechanisms. Generically, simple mechanisms converge to full efficiency as agents' risk aversion goes to infinity. Under risk neutrality, *ex-ante* optimal mechanisms are deterministic, and under risk aversion, they are not. Our results are suitable for applications.

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1 Introduction

Bilateral trade is a fundamental problem of economics. A unit of an indivisible commodity is to be traded between a seller and a buyer. The seller has a private cost of producing the good, and the buyer has a private valuation, these are traders' *types*. Traders may be risk averse, the general shape of their utility functions determines the *environment*, which is common knowledge.[1] A desirable model of this situation ought to be robust, that is, not too sensitive to the details of the specification of traders' information.[2] The problem is that if trade is voluntary, traders have incentives to misrepresent their private information, and efficient exchange is impossible. Robust mechanisms for the bilateral-trade problem were first discussed by Hagerty and Rogerson [1987] under the assumption of risk neutrality. In this paper, allowing for a wider class of environments, we define a suitable notion of constrained optimality for robust mechanism design and we provide optimality bounds for the allocations that can be achieved in equilibrium.

Our work thus brings two innovations. First, we introduce *ex-post* constrained efficiency as the optimality criterion which is congruent with robustness. Second, our method allows us to analyze risk-averse environments, and consequently allows for making comparative statics across different environments. Clearly, risk aversion plays an important role in many bilateral settings, such as wage bargaining or real-estate markets. In the absence of noncooperative theories, applied economists have used cooperative bargaining solutions to analyze such settings. We demonstrate that requiring robustness simplifies the mechanism-design problem, and allows for the analysis of general environments with risk aversion.

There are two aspects to the question of what are the second-best allocations that may be achieved in equilibrium. First, under robustness the traders need not know the distribution of types. Thus, the appropriate equilibrium constraints are the *ex-post* incentive and participation constraints. In an environment with private values (such as the present one), this observation is due to Ledyard [1978]. More recently, Bergemann and Morris [2005], Chung and Ely [2003], and Jehiel et al. [2005], provide foundational work for robust implementation.

While the first aspect of robustness determines the appropriate notion of incentive and participation constraints, the second aspect concerns the mechanism designer and determines the appropriate notion of constrained optimality. That is, when the mechanism designer does not know the distribution of traders' types he can only Pareto maximize traders' *ex-post* utility allocations, and the class of mechanisms that obtain is the class of *ex-post* constrained-efficient mechanisms, which we define.

[1]Myerson and Satterthwaite [1983] address this problem under the assumptions that agents are risk neutral, and that types are drawn independently from a distribution which is common knowledge.

[2]Wilson [1987] advocates a detail-free approach to mechanism design.

Under *interim* incentive and participation constraints, an analogous notion is the *ex-post* incentive efficiency, due to Holmstrom and Myerson [1983], and *ex-post* constrained efficiency is its natural extension for the setting with *ex-post* incentive and participation constraints. *Ex-post* constrained efficiency is defined via *ex-post* Pareto domination, where a mechanism in order to dominate some other mechanism, must allocate better utilities to *all* draws of types. In comparison, *ex-ante* domination requires that the mechanism dominate another mechanism *on average*. It is therefore easier to dominate a mechanism *ex ante* than it is *ex post*, i.e., there are many mechanisms that are *ex-post* Pareto incomparable, but once we take the averages over types we might be able to compare them. Thus, *ex-ante* constrained efficiency is a much stronger notion of constrained efficiency, but it depends on the distribution of traders' types.[3] In contrast, *ex-post* constrained efficiency is a distribution-free Paretian criterion, allowing for general statements about risk-averse environments, where utility is nontransferable.

In Section 3, we provide sufficient conditions for *ex-post* constrained efficiency, under *ex-post* incentive and participation constraints. In addition to these constraints, it suffices that the mechanism be *simple*, and that trade takes place with probability one when reports are the lowest-cost and the highest-valuation. Simplicity means that the mechanism can be described by two functions of traders' reports: a probability of transferring the object and the price at which to trade, conditional on the object being transferred. In an incentive-compatible simple mechanism, the traders have incentives to report truthfully as a result of a trade-off between the probability of trade and the price that they obtain. An example of a simple mechanism is a posted price, where the price is constant and the probability of trade is either 0 or 1. Under risk neutrality, all mechanisms are representable in a simple form. More precisely, for a given mechanism satisfying the incentive and participation constraints, we can find a simple mechanism satisfying incentive and participation constraints which gives the same utility allocation to *all* draws of traders' types.[4]

In general environments, not all mechanisms are simple. For instance, except for risk-neutral environments, randomizations over posted prices are neither simple nor *ex-post* constrained efficient. In general, under risk aversion a nonsimple mechanisms may satisfy incentive constraints, but when it is recast into a simple form the resulting simple mechanism need not satisfy the incentive constraints.[5]

[3]Much of the discussion in Holmstrom and Myerson[1983], relating *ex-post* incentive efficiency to other notions of efficiency, applies also to the present context with *ex-post* incentive and participation constraints, e.g. every *ex-ante* constrained efficient mechanism has to satisfy *ex-post* constrained efficiency, but the reverse need not be the case.

[4]Utility allocation is in general the expected utility to each draw of types, where the randomization is given by the randomization in the operation of the mechanism.

[5]Under risk aversion it is possible to reparametrize the utility allocation to both traders as arising from a price and probability of trade functions, but the resulting mechanism will in general fail to satisfy the incentive constraints.

In Section 4 we discuss risk neutral environments. We review the observation due to Hagerty and Rogerson [1987] that mechanisms satisfying *ex-post* incentive and participation constraints are payoff equivalent to lotteries over posted prices. Their original proof required strong technical assumptions; these restrictions limited the class of mechanisms under consideration and therefore precluded efficiency assessments. We prove that the equivalence holds in full generality. Hence we can establish that all *ex-post* constrained-efficient mechanisms, under *ex-post* incentive and participation constraints, are representable as lotteries over posted prices.

In section 5 we provide a characterization of simple mechanisms for constant relative risk-aversion environments. These are no longer representable by lotteries over posted prices. When traders become infinitely risk averse, the allocations generically converge to full *ex-post* efficiency. The intuition behind this is quite general. As the agents become more risk averse, the probability of implementing a bad outcome (in our case, no trade) in order to provide the agents with the right incentives, can get smaller and smaller. Comparing this to the Myerson and Satterthwaite [1983] result, the effect of risk aversion in the limit overrides the impossibility result even under the stronger *ex-post* incentive and participation constraints.[6]

We conclude our analysis with an example of *ex-ante* constrained efficiency. That example shows how the characterization of *ex-post* constrained efficiency can be used as a tool in the analysis of *ex-ante* welfare. Under risk neutrality, for a given type distribution, the *ex-ante* constrained-efficient mechanism is a posted price, while under risk aversion, it is a mechanism in which the trading price depends on traders' valuations. Assuming that in a world with stationary uncertainty, only *ex-ante* constrained-efficient exchanges should be observed, this provides a positive observation. In markets with large risk, relative to traders' wealth, we observe dispersed prices (correlated to traders' valuations), while in markets where risk is small, we observe posted pricing. Anecdotal evidence corroborating this observation is abundant: objects of small value are generally exchanged at posted prices, while in markets for objects with large values, such as real estate, the prices are generally negotiated; in markets of the underdeveloped world, where there is arguably more risk, many more goods are bargained at bazaars. In Section 6 we provide a short conclusion and discuss some extensions.

2 The problem

A seller, s, and a buyer, b, bargain over the price of an indivisible commodity. The payoff of trader i, $i = s, b$, from trading at a price $p \in [0, 1]$ is given by utility function

[6] Clearly, this implies that also in the Bayesian case (under *interim* incentive and participation constraints), as agents become infinitely risk averse, the constrained-efficient mechanisms tend to full *ex-post* efficiency.

$u_i(v_i, p) : [0,1] \times [0,1] \to R$, u_i is twice continuously differentiable and $u_i(p,p) = 0$. Traders obtain 0 if no trade and no transfers take place. Furthermore, $u_s(v_s, p)$ is increasing in p, decreasing in v_s, concave in each parameter, and satisfying the single-crossing condition $\frac{\partial^2 u_s}{\partial v_s \partial p} \leq 0$. Similarly, $u_b(v_b, p)$ is decreasing in p, increasing in v_b, concave in each parameter, and satisfying the single-crossing condition $\frac{\partial^2 u_b}{\partial v_b \partial p} \leq 0$. For instance, if $u_s(v_s, p) = u_s(p - v_s)$ and $u_b(v_b, p) = u_s(v_b - p)$, $u_i : [0,1] \to R, i = s, b$, are increasing, concave, and twice differentiable, then the above assumptions are satisfied. We denote $u = (u_s, u_b)$, and we call u the *environment*.

Parameters v_s and v_b are traders' private reservation values, or *types*. The interpretation is that v_s is the seller's cost of producing the good, and v_b is the buyer's valuation of the good. It is common knowledge that pairs of types $v = (v_s, v_b)$ are drawn from $[0,1] \times [0,1]$, according to *some* continuous joint distribution function F, with a strictly positive density f on $[0,1] \times [0,1]$.[7] We stress that congruent with the notion of robustness, F need not be common knowledge, so that traders may have different beliefs about F, and different beliefs about the beliefs of the other trader and so on. We abstract from such considerations by simply assuming that the details of F are unknown to the traders, and we use the appropriate equilibrium notion consistent with this assumption.

A *direct revelation mechanism* (from now on a *mechanism*) is a game form, mapping traders' reports of their reservation values into outcomes.[8] Denote by p_s the payment received by the seller and by p_b the price charged to the buyer when the object is transferred, and let ω_{NT} denote the no trade and no transfers outcome. We assume that outcomes have to be feasible, that is, no outcome should require any subsidies *ex post* so that $p_s \leq p_b$; ω_{NT} is always feasible. We denote the vector of traders' reports by $\tilde{v} = (\tilde{v_s}, \tilde{v_b})$. Given traders' reports, an outcome is given by a lottery $\mu[\tilde{v}]$ over the feasible set, $\{(p_s, p_b) \mid p_s \leq p_b\} \cup \{\omega_{NT}\}$. Note that $\mu[\tilde{v}]$ is the lottery which the traders face *ex post*, after having reported their types. A mechanism m is thus a collection of lotteries $m = \{\mu[\tilde{v}] \mid \mathbf{supp}(\mu[\tilde{v}]) \subset \{(p_s, p_b) \mid p_s \leq p_b\} \cup \{\omega_{NT}\}, \tilde{v} \in [0,1]^2\}$, a lottery $\mu[\tilde{v}]$ for each vector of reports $\tilde{v}$.

Given a mechanism m, when traders report $\tilde{v}$, the expected utility of agent i with the reservation value v_i is

$$U_i^m(\tilde{v}, v_i) = E_{\mu[\tilde{v}]}\{u_i(v_i, p_i)\}, i = s, b,$$

where E_μ denotes the expectation with respect to the measure μ. We slightly abuse

[7] We could generalize our analysis to environments where vector v is drawn from $[\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}]$, $\underline{v} < 1/2 < \overline{v}$. This is equivalent to the requirement that F has support on $[\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}]$. Note also that common knowledge of the support of F is sufficient for our analysis, but it might not be necessary.

[8] We emphasize that while the revelation principle does not apply directly, it is well known that in the present setting with private values a version of the revelation principle does hold. See Ledyard [1978] and more recently Bergemann and Morris [2005] for more details.

the notation and denote by $U_i^m(v)$ the expected utility of agent i, $i = s, b$, when she reports truthfully, $U_i^m(v) = U_i^m(v, v_i)$. We stress that the expectation operator E_μ has nothing to do with the distribution of traders' types: $U_i^m(\tilde{v}, v_i)$ is the *ex-post* expected payoff that player i obtains in mechanism m when the reports are $\tilde{v}$. Measure μ refers to the randomization over prices for a given report $\tilde{v}$, which is one point in the type space.

Apart from feasibility, we consider mechanisms which are *ex-post* individually rational (XPIR) and *ex-post* incentive compatible (XPIC). We thus require that trade always be voluntary *ex post*, and that reporting the reservation values truthfully is an *ex-post* equilibrium. As we are considering direct-revelation mechanisms in an environment with private values, this is equivalent to requiring that reporting reservation values truthfully is a dominant-strategy equilibrium of the game form defined by the direct-revelation mechanism.

(XPIR) Ex-post Individual Rationality.
A mechanism m is *ex-post individually rational* if
$supp(\mu[\tilde{v}]) \subset \{(p_s, p_b) \mid \tilde{v}_s \leq p_s \leq p_b \leq \tilde{v}_b\} \cup \{\omega_{NT}\}$, $\forall \tilde{v} \in [0,1]^2$.

(XPIC) Ex-Post Incentive Compatibility.
A mechanism m is *ex-post incentive compatible* if

$$U_i^m(v) \quad \geq \quad U_i^m(\tilde{v}_i, v_j, v_i) \, \forall v_i \forall v_j \forall \tilde{v}_i, i = s, b, j \neq i.$$

It is well known and immediate to prove that XPIC implies monotonicity of utility allocations to the traders.

Lemma 1. Let m be XPIC. Then $U_s^m(v)$ is strictly decreasing in v_s, whenever $U_s^m(v) > 0$; and $U_b^m(v)$ is strictly increasing in v_b, whenever $U_b^m(v) > 0$.

Proof: We provide the proof for the seller. Let $U_s^m(v_s, v_b) > 0$, for some $0 < v_s < v_b$ and let $v_s' < v_s$. Then it must be that $\mu[v]$ assigns a positive probability to some feasible prices, so that by strict monotonicity of u_s in v_s, we have $U_s^m(v_s', v_b, v_s) > U_s^m(v_s, v_b, v_s)$. Hence, by XPIC,

$$U^m(v_s', v_b, v_s') \geq U^m(v_s', v_b, v_s) > U^m(v_s, v_b, v_s).$$

■

In addition to monotonicity, another technical property that is important is the absolute continuity of payoffs at the truthful reports. We call mechanisms that satisfy this property *regular*. That is, we say that m is *regular*, if $U_i^m(v_i, v_j)$ is absolutely continuous with respect to v_i, for every v_j. In the sequel we will restrict attention to regular mechanisms. However, we remark that for an important class of environments this requirement is innocuous since the stronger property of Lipschitz continuity is implied by XPIC:

Lemma 2. Let m be XPIC. If $u_s(v_s,p) = u_s(p-v_s)$ and $u_b(v_b,p) = u_s(v_b-p)$, with $u_i'(0) < \infty$, then $U_i^m(v_i,v_j)$ is Lipschitz in v_i $\forall v_j, i = s,b, j \neq i$.

Proof: See Appendix A.■

From now on, whenever we write *XPIRIC mechanism* we mean a direct revelation mechanism satisfying XPIR and XPIC, and regularity. A very simple example of XPIRIC mechanism is a *posted price*.

Example 3. Consider an environment such that u_i is monotone in the amount of surplus obtained by i. A posted price is defined by the price, which is deterministic and is independent of the traders' reports. Once the traders observe the price, they trade if they both find it optimal to do so. Formally,

$$\pi(v) = p \in [0,1], \varphi(v) = \begin{cases} 1 \text{ if } v_b \geq p \geq v_s, \\ 0 \text{ otherwise.} \end{cases}$$

In a posted price, it is clearly optimal for each trader to report his valuation truthfully, regardless of the report of the other trader, so that XPIC holds; XPIR is obviously satisfied.

While a posted price is an example of a mechanism where the verification of the incentive and participation constraints is trivial, the following provides an example of a slightly more involved mechanism. In particular, the verification of incentive and participation constraints is sensitive to the environment. We will analyze similar mechanisms more thoroughly in the subsequent sections.

Example 4. Let the environment be *symmetric*, specified by utility functions (u_s,u_b), $u_i : [0,1]^2 \to R$, where $u_s(v_s,p) = \bar{u}(p-v_s)$, $u_b(v_b,p) = \bar{u}(v_b-p)$. Define the mechanism m by the following collection of lotteries. For reports $\tilde{v}$, s.t. $\tilde{v}_b > \tilde{v}_s$, let $\mu[\tilde{v}]$ be given by a binary lottery, allocating probability $\varphi(\tilde{v}) = \frac{\bar{u}(\tilde{v}_b-\tilde{v}_s)}{\bar{u}(1)}$ to trade at a price $\pi(\tilde{v}) = \frac{1}{2}(\tilde{v}_s+\tilde{v}_b)$, and a probability $1-\varphi(\tilde{v})$ to ω_{NT}. For reports $\tilde{v}$ s.t. $\tilde{v}_b \leq \tilde{v}_s$ let $\varphi(\tilde{v}) = 0$, and $\pi(\tilde{v}) = \frac{1}{2}(\tilde{v}_s+\tilde{v}_b)$.

Clearly, m satisfies XPIR. To see that XPIC also holds take for instance the seller of type v_s, who faces the following optimization problem,

$$\tilde{v}_s \in \arg\max_{\tilde{v}_s \in [0,1]} \varphi(\tilde{v}_s,\tilde{v}_b)\bar{u}\left(\pi(\tilde{v}_s,\tilde{v}_b) - v_s\right), \forall \tilde{v}_b.$$

Taking derivatives, it is immediate to verify that independently of $\tilde{v}_b$, $\tilde{v}_s = v_s$ is the unique maximizer of this optimization, given the above specification of φ and π. This verifies that XPIC is also satisfied.

2.1 Efficiency and *ex-post* constrained efficiency

Next, we define the efficiency requirements. *Ex-post* efficiency is a standard requirement, albeit a very strong one.

(EFF) Ex-Post Efficiency.
A mechanism m is *ex-post efficient* if the allocation $(U_s^m(v), U_b^m(v))$ is Pareto-optimal for each $v \in [0,1]^2$.

Example 3, *continued.* No posted price satisfies EFF, since in a posted price it can always happen *ex post* that either $p > v_b > v_s$ or $v_b > v_s > p$.

XPIRIC and EFF mechanisms do not exist. The following proposition is a simple extension of the Myerson and Satterthwaite [1983] impossibility result to the present setup. The *ex-post* incentive and participation constraints are stronger than the *interim* constraints considered in Myerson and Satterthwaite [1983]. For that reason, the proof of the impossibility result is very simple in the present setup. Note that the impossibility result stated here is general and relies only on u being monotonic; Myerson and Satterthwaite [1983] result requires risk neutral traders.

Proposition 5. There does not exist an XPIRIC bilateral-trade mechanism satisfying EFF.

Proof: Let $m = \{\mu[v]; v \in [0,1]^2\}$ be an XPIRIC and EFF mechanism. We show that this is impossible. For $v \in [0,1]^2$ s.t. both traders are risk neutral on $\mathbf{supp}(\mu[v])$, define $\bar{\pi}(v) = E_{\mu[v]}[p]$. Clearly, for all such v, $U_i^m(v) = u_i(\bar{\pi}(v), v_i)$. Next, for all $v \in [0,1]^2$, s.t. at least one trader has a strictly concave utility function on $\mathbf{supp}(\mu[v])$, it has to be that $\mathbf{supp}(\mu[v])$ is a singleton. Otherwise the allocation under lottery $\mu[v]$ would not be Pareto efficient. Denote the price at which trade occurs by $\bar{\pi}(v)$, and again $U_i^m(v) = u_i(\bar{\pi}(v), v_i)$, for all such v. It is immediate to check that monotonicity of $U_i^m, i = s, b$, implies monotonicity of π. Thus, by Lemma 1, $\bar{\pi}(v)$ is increasing in both v_s and v_b. By XPIR, it must be that $\bar{\pi}(x,x) = x, \forall x \in [0,1]$. Now take a $v = (v_s, v_b), v_s < v_b$. If $\bar{\pi}(v) > v_s$, then b would misreport to $v_b' = v_s$, so XPIC for b would be violated. If $\bar{\pi}(v) < v_b$, then s would misreport to $v_s' = v_b$, a contradiction.$\blacksquare$

Since EFF is not possible we consider XPIRIC mechanisms that attain constrained-efficient allocations. The constrained-efficiency criterion that we propose is the *ex-post constrained efficiency.* This notion is an extension of the *ex-post* incentive efficiency, due to Holmstrom and Myerson [1983].

(XPCE) Ex-Post Constrained Efficiency.
Denote the set of incentive and participation constraints by $\mathcal{IP}$ (these could be either *ex-post*, *interim*, or any other set of participation and incentive constraints). A

mechanism m, satisfying $\mathcal{IP}$, is *ex-post dominated, under* $\mathcal{IP}$, by another mechanism m', $m' \succ_{xp|\mathcal{IP}} m$, if m' satisfies $\mathcal{IP}$, and

$$U_s^{m'}(v, v_s) \geq U_s^m(v, v_s) \text{ and } U_b^{m'}(v, v_b) \geq U_b^m(v, v_b), \forall v,$$

with a strict inequality for an open set of v's, for at least one of the traders. A mechanism m is *ex-post constrained efficient* under $\mathcal{IP}$, if there does not exist a mechanism m' s.t. $m' \succeq_{xp|\mathcal{IP}} m$. We call XPCE mechanisms, under XPIRIC, *cexpiric* mechanisms.

The notion of *ex-post* constrained efficiency is tailored to our assumption that the joint distribution of traders' valuations has a full support and is continuous (regardless of the exact shape of the distribution). The requirement that the strict inequality hold for an open set of types is then equivalent to requiring that the event in which at least one player is strictly better off have a nonzero probability. Equivalently we could require that for at least one trader, the Lebesgue measure of the set of types that are strictly better off must be positive.

Example 3, *continued.* Let $\bar{p}$ and $\bar{\bar{p}}$ be two posted prices, $0 \leq \bar{p} < \bar{\bar{p}} \leq 1$. Then neither $\bar{\bar{p}} \succ_{xp|\mathcal{IP}} \bar{p}$ nor the other way around. To see for instance the former, observe that under $\bar{p}$ the draws of types v s.t. $v_s < \bar{p} < v_b$ all obtain a strictly positive utility, while under $\bar{p}$ these pairs of traders obtain a 0 utility.

When $\mathcal{IP}$ are the *interim* incentive and participation constraints, this constrained efficiency notion is equivalent to the *ex-post* incentive efficiency as defined by Holmstrom and Myerson [1983]. Per se, XPCE does not depend on the specification of the distribution of traders' types, so that this is an optimality criterion that is suitable for robustness. Moreover, since it is a Paretian criterion, no assumptions are made on the interpersonal utility comparisons, which is important for the environments with risk aversion (i.e., environments with nontransferable utility).

Clearly,

$$\emptyset = \{m \mid m \text{ XPIRIC and EFF }\} \subset \{m \mid m \text{ cexpiric}\},$$

where the first equality follows from Proposition 5. The requirements under XPIR and XPIC defined above are the strongest participation and incentive-compatibility criteria, but XPCE is the weakest constrained-efficiency notion; XPIR and XPIC are stronger than their *interim* analogs, while XPCE is weaker than *interim* constrained efficiency, which in turn is weaker than the *ex-ante* constrained efficiency. In other words, regardless of what is specified by $\mathcal{IP}$, the sets of the *ex-ante* and the *interim* constrained-efficient mechanisms are supersets of the EFF mechanisms, and subsets of the XPCE mechanisms.

3 Simple mechanisms and constrained efficiency

Next we examine conditions that assure *ex-post* constrained efficiency, under *ex-post* incentive and participation constraints. We start by a very simple sufficient condition.

Proposition 6. Posted prices are cexpiric.

Proof: Let m be a posted price, given by some $p^* \geq 0$. That m satisfies XPIRIC is obvious. We show that there are no XPIRIC mechanisms which *ex-post* dominate posted prices.

Suppose there exists a m' s.t. $m' \succeq_{xp} p^*$, (we use p^* to refer both to the mechanism m and to the posted price). Since on the set $\{v \mid v_s \leq p^* \leq v_b\}$ the allocation under p^* is Pareto optimal, the allocation under m' has to coincide with the allocation under p^* on that set. In particular, on the line segments $v_s = p^*$ and $v_b = p^*$, m' is identical to p^*, otherwise the XPIR constraints for m' would be violated. Thus, by monotonicity of $U_i^{m'}$ w.r.t. v_i, $U_s^{m'}(v) = 0$ for $v_s > p^*$, and $U_b^{m'}(v) = 0$ for $v_b < p^*$. Since $m' \succeq_{xp} p^*$, by definition of *ex-post* constrained efficiency, there exists either an open rectangle $\Gamma \subset \{v \mid p^* \geq v_b > v_s\}$ s.t. $U_s^{m'}(v) > 0$ for $v \in \Gamma$, or an open rectangle $\Gamma'^* \leq v_s < v_b\}$ s.t. $U_b^{m'}(v) > 0$ for $v \in \Gamma'$. Both of these two cases are treated in exactly the same way so we consider the first possibility. Since $U_b^{m'}(v) = 0$ for $v_b < p^*$ (by monotonicity of $U_b^{m'}$ and $U_b^{m'}(v_s, p^*) = 0$), it is clear that $U_b^{m'}(v) = 0$ for $v \in \Gamma$, and since for $v \in \Gamma$, $U_s^{m'}(v) > 0$, it must be that on Γ, m' is a mechanism that for the buyer randomizes between one price $\pi'(v) = v_b$ and ω_{NT}, and the probability on $\pi'(v) = v_b$ must be positive. Denote this probability by $\varphi_b'(v)$. So fix a $\bar{v} \in \Gamma$. Clearly, $v_b = p^*$ has incentives to report $\bar{v}_b$ instead of p^* since $\varphi_b'(\bar{v})u_b(\bar{v}_b, p^*) > 0 = U_b^{m'}(v_s, p^*)$, a contradiction.$\blacksquare$

On a more abstract level, one can think of a mechanism as an assignment of feasible *ex-post* utility payoffs. Under risk neutrality, the standard parametrization of these payoffs is obtained by specifying, at each vector of reports, the probability of transferring the object, and the expected monetary transfer between the traders. Such parametrization is without loss of generality only under risk neutrality, if XPIRIC hold. A slightly different parametrization is more convenient here. In general environments, we parametrize traders' expected utilities by the *probability of trade* and the *price at which to trade, conditional on trade taking place* (both are functions of traders' reports). As we mentioned above, under risk neutrality, this parametrization is equivalent to the standard one. In general environments, we call the mechanisms that can be parametrized in this way *simple mechanisms*.[9]

[9] Clearly, if $\mathcal{IP}$ are not imposed then such parametrization is always without loss of generality. However, in a general environment, when a mechanism m satisfying XPIRIC, but which is not *simple*, is reparametrized into a simple form, it may no longer satisfy XPIRIC.

A *simple mechanism* m is a mechanism where each $\mu[\tilde{v}]$ is a binary lottery between trade at a specific price and ω_{NT}. A simple mechanism m is represented by a pair of functions $(\pi, \varphi) : [0,1]^2 \to [0,1]^2$. Given traders' reports, $\pi(\tilde{v})$ is the price at which the traders trade, $\varphi(\tilde{v})$ is the probability of trading at that price, and $1 - \varphi(\tilde{v})$ is the probability of ω_{NT}. The mechanisms introduced in examples 3 and 4 were both simple. For instance, in a posted price $\bar{p}$, $\pi(v) = \bar{p}, \forall v \in [0,1]^2$; $\varphi(v) = 1$ if and only if $v_b \geq \bar{p} \geq v_s$, and $\varphi(v) = 0$ otherwise.

As we have shown in Proposition 6, one (very strong) sufficient condition for a mechanism to be cexpiric is that it is a posted price. We can relax this condition considerably. Under a mild assumption on the utility functions, simple mechanisms satisfying XPIRIC, and such that the lowest-cost seller and the highest-valuation buyer trade *ex-post* with certainty are cexpiric.

Proposition 7. Let $u_s(v_s, p) = u_s(p - v_s)$ and $u_b(v_b, p) = u_b(v_b - p)$. Assume $u_i''(x) < 0$, $\forall x \in [0,1]$ for at least one i, and $u_i'''(x) \geq 0$ for $i = s, b$. If a simple mechanism $m = (\pi, \varphi)$ is XPIRIC and $\varphi(0,1) = 1$, then m is cexpiric.

Proof: See Appendix A.■

In the following example, we provide two mechanisms. One mechanism is simple, the other is not. In this example, the simple mechanism is a special case of the simple mechanism from Example 4. It satisfies the assumptions of Proposition 7 and it *ex-post* dominates the nonsimple mechanism.[10]

Example 8. Let $u_s(p, v_s) = (p - v_s)^\gamma, u_s(p, v_b) = (v_b - p)^\gamma, \gamma \in (0,1]$. Consider the following two mechanisms. Let m be simple and given by $\pi(\tilde{v}) = \frac{\tilde{v}_s + \tilde{v}_b}{2}$ and $\varphi(\tilde{v}) = \max\{0, (\tilde{v}_b - \tilde{v}_s)^\gamma\}$. In Example 4 we checked that m is XPIRIC. Next, let $\bar{m}$ be given by lottery $F_p \equiv U[0,1]$ over posted prices, where $U[0,1]$ denotes the uniform distribution over $[0,1]$. In other words, $\bar{m}$ is a mechanism where the price is drawn randomly from a uniform distribution, and the traders trade if it is individually rational for both - so XPIR is satisfied. XPIC also holds for $\bar{m}$ since traders do not affect the price draw with their reports, and by misreporting they can only be worse off. When traders are risk neutral, i.e., $\gamma = 1$, $U_i^m(v) = U_i^{\bar{m}}(v), \forall v, i = s, b$, so that m and $\bar{m}$ are equivalent in the sense that they both satisfy XPIRIC, and the allocation to every draw of types is the same. When $\gamma < 1$, $\bar{m}$ is not simple, and it is *ex-post* dominated by m. We return to this in Section 5.

3.1 Differentiable mechanisms and the first-order conditions

If in a mechanism m the expected utilities of the traders are differentiable, then the XPIC can be specified as a first-order condition (FOC). In this subsection, we show

[10]Our conjecture is that under the above assumptions on the environment, simplicity is also necessary for *ex-post* constrained efficiency. Insofar we have been unable to prove this.

that if a simple mechanism is differentiable, then this FOC is necessary and sufficient, so that all simple differentiable XPIRIC mechanisms are given as all possible differentiable solutions (π, φ) to the FOC.

Given a mechanism m, we denote by S^m the set of types where both traders obtain a strictly positive expected utility under truthful reports. When $m = (\pi, \varphi)$, $S^{\pi,\varphi}$ can be written as

$$S^{\pi,\varphi} = \{v \mid v \in [0,1] \times [0,1], \varphi(v) > 0, v_s < \pi(v) < v_b\}.$$

(DIFF) Differentiability. A mechanism m is *differentiable* if $U_i^m(v)$ are differentiable on S^m.

We remark that a simple XPIRIC mechanism (π, φ) is differentiable if and only if π and φ are both differentiable, which follows from the Implicit Function Theorem and the fact that XPIC implies $U_i^m(v)$ is strictly monotonic in v_i on S^m.

Proposition 9. A simple and DIFF mechanism $m = (\pi, \varphi)$ is XPIRIC if and only if, $\forall v \in S^{\pi,\varphi}$,

$$\begin{array}{c} \frac{\partial \varphi(v)}{\partial v_s} u_s(\pi(v), v_s) = -\varphi(v) \frac{\partial u_s(\pi(v), v_s)}{\partial p} \frac{\partial \pi(v)}{\partial v_s}, \\ \frac{\partial \varphi(v)}{\partial v_b} u_b(\pi(v), v_b) = \varphi(v) \frac{\partial u_b(\pi(v), v_b)}{\partial p} \frac{\partial \pi(v)}{\partial v_b}. \end{array} \tag{1}$$

Proof: We derive the FOC for the seller. It is necessary that

$$\frac{\partial U_s^m(v, v_s')}{\partial v_s'} \mid_{v_s' = v_s} = 0,$$

which gives the desired condition. For sufficiency see Appendix A.■

The interpretation of this FOC is that the traders are provided with the correct incentives by a marginal trade-off between the price and the probability of trade.

4 Risk neutrality

In what follows we will argue that when traders are risk neutral, the set of cexpiric mechanisms is equivalent to the set of probability distributions over posted prices, in terms of utility allocations to the traders.

A *distribution over posted prices* is a mechanism given by some increasing function $F_p : [0,1] \to [0,1]$, with $F_p(0) = 0$ and $F_p(1) \le 1$. The posted price p is drawn at random according to F_p, independently from trader's reports, and the traders then trade at p if and only if trading at p is individually rational for both of them. A *probability distribution over posted prices* is a distribution over posted prices such that $F_p(1) = 1$.

Suppose that F_p is a *probability* distribution over posted prices. Then in risk-neutral environments, we can represent F_p as a simple mechanism by defining $\varphi_{F_p}(\tilde{v})$ as the mass under F_p between $\tilde{v}_s$ and $\tilde{v}_b$, whenever $\tilde{v}_s < \tilde{v}_b$. The price, $\pi_{F_p}(\tilde{v})$ is defined as expected price, under F_p conditional on trade taking place. Observe then that since F_p is a probability distribution, $\varphi(0,1) = 1$ - i.e., the lowest-cost seller and the highest-valuation buyer trade with probability 1, regardless of the specification of F_p. As in Example 8, it is easy to verify that every distribution over posted prices satisfies XPIRIC. Moreover, under risk neutrality agents' incentives do not change if we represent F_p as φ_{F_p}, π_{F_p}. Thus, under risk neutrality, by Proposition 7, every probability distribution over posted prices is cexpiric.

Our next result establishes the payoff equivalence of distributions over posted prices and XPIRIC mechanisms

Proposition 10. For $u_i(x) = x, i = s, b$, a mechanism (π, φ) is XPIRIC if and only if there exists a distribution function over posted prices F_p, such that

$$\pi(v) = E_{F_p}[\omega \mid \omega \in (v_s, v_b)]$$

$$\varphi(v) = \max\left\{F_p(v_b) - F_p(v_s), 0\right\}.$$

This equivalence was previously established by Hagerty and Rogerson [1987] under strong technical restrictions (i.e. either (φ, π) are twice continuously differentiable, or the image of φ is $\{0,1\}$). Proposition 10 is a substantial generalization of their result, since our proof relies only on properties directly implied by XPIRIC.

To prove Proposition 10 we rely on Lemma 11. This Lemma is technical but very important; it supplies a crucial separability property that allows the link of XPRIC mechanisms to distribution functions over posted prices without the need to impose technical requirements beyond those directly implied by XPIRIC. As the proof of Lemma 11 is long and involved but not specially illuminating for our analysis we present it in Appendix B.

Lemma 11. Consider a function $\varphi(v_s, v_b)$, bounded, increasing in v_s, decreasing in v_b, and nonnegative, for $(v_s, v_b) \in [0,1]^2$. Let $\varphi(v_s, v_b)$ satisfy,

$$\varphi(v_s, v_b) = \frac{1}{v_b - v_s} \int_{v_s}^{v_b} (\varphi(\tau, v_b) + \varphi(v_s, \tau)) d\tau, \forall (v_s, v_b) \in [0,1]^2. \qquad (2)$$

Then $\varphi(v_s, v_b) = \tilde{\varphi}(v_b) - \tilde{\varphi}(v_s), \forall v_b \geq v_s$, where $\tilde{\varphi}(.)$ is some increasing function.

The proof of Proposition 10 follows. The crucial step is to show that XPRIC implies the conditions of Lemma 11.

Proof of Proposition 10: First, a distribution $F_p(.)$ over posted prices satisfies XPIRIC, since every posted price is XPIRIC and F_p is independent of traders' reports. The *simple* representation of the mechanism given by F_p is

$$\varphi(v) = \max\{F(v_b) - F(v_s), 0\}$$

and

$$\pi(v) = E_{F_p}[\omega \mid \omega \in (v_s, v_b)],$$

i.e., expected payoffs are the same as those generated under F_p (it is very easy to verify this).

For the converse, take an XPIRIC (π, φ). It is enough to show that $\varphi(v) = \varphi(0, v_b) - \varphi(0, v_s)$ since we can then define $F_p(\omega) = \varphi(0, v_b)$ and it follows immediately that $\pi(v) = E_{F_p}[\omega \mid \omega \in (v_s, v_b)]$.

XPIRIC implies that $\varphi(.)$ is nonincreasing in v_s and nondecreasing in v_b. By Lemma 1, $U_i^{\pi,\varphi}(v)$ is monotonic in v_i, whenever $U_i^{\pi,\varphi}(v)$ is strictly positive. Take the seller and let $v_s' > v_s$. By XPIC,

$$\varphi(v_s, v_b)(\pi(v_s, v_b) - v_s) \geq \varphi(v_s', v_b)(\pi(v_s', v_b) - v_s), \text{ and}$$

$$\varphi(v_s', v_b)(\pi(v_s', v_b) - v_s') \geq \varphi(v_s, v_b)(\pi(v_s, v_b) - v_s').$$

By subtracting first the rhs, and then the lhs of the second inequality from the first inequality, we obtain

$$\varphi(v_s, v_b)(v_s - v_s') \geq U_s^{\pi,\varphi}(v_s, v_b) - U_s^{\pi,\varphi}(v_s', v_b) \geq \varphi(v_s', v_b)(v_s - v_s').$$

Thus, φ is weakly decreasing in v_s, and by Lemma 2 it is absolutely continuous. Hence, it is an integral of its derivative. Again, by the above inequalities, $\frac{\partial U_s^{\pi,\varphi}(z,v_b)}{\partial v_s} = \varphi(v)$, whenever this derivative exists. Thus, $U_s^{\pi,\varphi}(v)$ can be expressed as

$$U_s^{\pi,\varphi}(v) = \int_{v_s}^{v_b} \frac{\partial U_s^{\pi,\varphi}(z, v_b)}{\partial v_s} dz = \int_{v_s}^{v_b} \varphi(z, v_b) dz.$$

Similarly, we obtain $U_b^{\pi,\varphi}(v) = \int_{v_s}^{v_b} \varphi(v_s, z) dz$, and adding the two equations yields

$$\varphi(v) = \frac{1}{v_b - v_s} \int_{v_s}^{v_b} (\varphi(v_s, z) + \varphi(z, v_b)) dz, \forall v \in [0,1]^2.$$

The claim now follows from Lemma 11. ■

With the payoff equivalence result established in full generality, we can draw definite conclusions regarding constrained efficiency. Because no types can trade with a probability higher than 1, $F_p(1) \leq 1$ is a feasibility restriction on the distributions. If a distribution over posted prices is not a probability distribution then it is *ex-post* dominated by some probability distribution, simply by multiplying the distribution so that its mass becomes 1. On the other hand, a probability distribution over posted prices is not *ex-post* dominated by another probability distribution over posted prices, the proof of which is the same as the proof that two posted prices do not *ex-post* dominate each other. It is then a straightforward corollary of Proposition 10 that, under risk neutrality, if a mechanism is cexpiric, then it must be representable as a probability distribution over posted prices.

Corollary 12. In the risk-neutral environment, a mechanism m is cexpiric if and only if m can be represented as a probability distribution over posted prices.

We remark that by Corollary 12, the differentiable cexpiric mechanisms under risk neutrality are given simply by continuously differentiable probability distributions F_p over posted prices. Under risk neutrality, differentiable cexpiric mechanisms are therefore generic within the class of cexpiric mechanisms. Nonetheless, if the distribution of types were known, then the *ex-ante* optimal XPIRIC mechanism under risk neutrality is a degenerate distribution over posted prices (see Section 5.1).

5 Constant relative risk-aversion

In this section, we analyze symmetric *constant relative risk aversion* (CRRA) environments. CRRA utility functions are specified by $u_i(x) = x^\gamma, \gamma \in (0,1]$. Note that when $\gamma = 1$ this is the standard risk-neutral environment, and as γ tends to 0, risk aversion tends to infinity.

We will explicitly compute all simple XPIRIC mechanisms. This characterization will be used to show that in a sequence of symmetric environments, when relative risk aversion of traders converges to ∞ point-wise, every simple cexpiric m, satisfying $S^m = \{v \mid v_s < v_b\}$, converges to *ex-post* efficiency (EFF).

Proposition 13. Let $u_i(x) = x^\gamma$ for $\gamma \in [0,1], i = s, b$. Then a simple mechanism $m = (\pi, \varphi)$ is cexpiric if and only if

$$\varphi(v) = \begin{cases} \left(\int_{v_s}^{v_b} dF(z)\right)^\gamma, & \text{if } v_b \geq v_s, \\ 0, & \text{otherwise,} \end{cases}$$

$$\pi(v) = \frac{1}{F(v_b) - F(v_s)} \int_{v_s}^{v_b} x dF(x), \text{ if } v_s < v_b,$$

(and $\pi(v) = v_s$, for $v_s \geq v_b$), for some probability distribution $F : [0,1] \to [0,1]$.

Proof: For $u_i(x) = x^\gamma, i = s, b$, wherever (π, φ) are differentiable, the first-order conditions (1) are,

$$\begin{aligned} &\tfrac{\partial\varphi(v)}{\partial v_s}(\pi(v) - v_s) + \gamma\tfrac{\partial\pi(v)}{\partial v_s}\varphi(v) = 0, \\ &\tfrac{\partial\varphi(v)}{\partial v_b}(v_b - \pi(v)) - \gamma\tfrac{\partial\pi(v)}{\partial v_b}\varphi(v) = 0. \end{aligned} \tag{3}$$

By setting $\varphi(v) = \bar{\varphi}(v)^\gamma$ we obtain exactly the same system of equations for $(\pi, \bar{\varphi})$ as under risk neutrality, and the claim follows from Proposition 10. ■

The following corollary is immediate.

Corollary 14. For $u_i(x) = x^\gamma, \gamma \in (0,1), i = s, b$, no lottery over posted prices is cexpiric. Conversely, a simple mechanism m that is cexpiric and is not a posted price is not representable by a lottery over posted prices.

Observe that Proposition 13 implies that every mechanism m, with $S^m = \{v \mid v_s < v_b\}$, satisfies the property that whenever traders become infinitely risk averse, the allocation converges to full efficiency. Under risk neutrality, such mechanisms are precisely probability distributions over posted prices with a full support.

5.1 *Ex-ante* optimality

We provide an example to illustrate the usefulness of the characterization in Proposition 13 in order to make statements about the *ex-ante* constrained-efficient mechanisms under risk aversion. We make two remarks. First, in order to perform *ex-ante* welfare analysis, one has to know the type distribution. The interpretation in the context of robustness is that this is a positive observation: if there is an underlying distribution of types, and we expect to observe only the constrained-efficient mechanisms, then *ex-ante* constrained efficiency is an appropriate notion. As we mentioned before, this class is a subclass of cexpiric mechanisms.

Second, we only proved that if incentive and participation constraints are met and trade assured for the maximum valuation and minimum cost pair, simplicity is sufficient - we did not prove that it is necessary. Thus, a nonsimple *ex-post* constrained efficient mechanism may exist, and it may be that such mechanism is *ex-ante* optimal.What the example shows is that necessarily, when traders are risk averse, the *ex-ante* optimal mechanism is not deterministic (trade may happen with positive probability not equal to 1). Namely, among the simple mechanisms the *ex-ante* optimal one is a lottery, and the only deterministic mechanisms are posted prices.[11]

We assume that the traders' types are i.i.d., uniformly distributed on $[0,1]$, so that $f(v_s, v_b) = 1, \forall v_s, v_b \in [0,1]$, where $f(.,.)$ is the density of F, the traders' distribution of types. To keep things simple we look at a utilitarian *ex-ante* social welfare function,

$$W^m = \int_{v_s \in [0,1]} \int_{v_b \in [0,1]} \left(U_s^m(v_s, v_b) + U_b^m(v_s, v_b)\right) f(v_s, v_b) dv_b dv_s,$$

[11] Another way to view this result is in terms of linear programing. Solving for the *ex-ante* optimal mechanism under risk neutrality is to solve a linear program on the convex set of cexpiric mechanisms, so that it is not surprising that a solution is generically a "corner" of this set, i.e., a posted price. When traders are risk averse, the *ex-ante* optimization is no longer a linear program, and the optimal mechanisms are more interesting.

where m is a mechanism.

The problem of designing the *ex-ante* optimal simple XPIRIC mechanism can be written as

$$\begin{array}{l} \max_m W^m \\ \text{s.t. } m \text{ XPIRIC and simple.} \end{array} \tag{4}$$

Every m which is *ex-ante* constrained efficient has to be cexpiric. Hence, it is in expression (4) enough to optimize over all cexpiric mechanisms. By Proposition 13 and since $f(.,.) \equiv 1$, the problem (4) can be written as

$$\max_{F_p} \int_0^1 \int_0^1 \left(\varphi(v_s, v_b)\left[(\pi(v_s, v_b) - v_s)^\gamma + (v_b - \pi(v_s, v_b))^\gamma\right]\right) dv_b dv_s,$$

where $\varphi(v_s, v_b) = (\max\{F_p(v_b) - F_p(v_s), 0\})^\gamma$ and $\pi(v_s, v_b) = E_{F_p}[p \mid v_s \leq p \leq v_b]$. This can be rewritten as

$$\max_{f_p} \int_0^1 \int_{v_s}^1 \left[\left(\int_{v_s}^{v_b} (t - v_s) f_p(t) dt\right)^\gamma + \left(\int_{v_s}^{v_b} (v_b - t) f_p(t) dt\right)^\gamma\right] dv_b dv_s.$$

Denoting $G(t) = \int_0^t F_p(\tau) d\tau$, and letting

$$\nu(v_s, v_b) = \left[G(v_b) - G(v_s)\right](v_b - v_s),$$

we can rewrite the above expression (integrate by parts each of the two innermost integrals and compute the appropriate derivatives) as

$$\max_{\nu} \int_0^1 \int_{v_s}^1 \left[\left(-2\frac{\nu(v_s, v_b)}{v_b - v_s} + \frac{\partial \nu}{\partial v_b}\right)^\gamma + \left(2\frac{\nu(v_s, v_b)}{v_b - v_s} + \frac{\partial \nu}{\partial v_s}\right)^\gamma\right] dv_b dv_s \tag{5}$$

The maximization problem (5) is a manageable optimization problem. We can in principle compute its solutions, using the calculus of variations. Except when $\gamma = 1$, we cannot compute the solutions in closed form. When $\gamma = 1$ the problem simplifies substantially since only the terms involving the derivatives of ν remain. It is then straightforward to compute that the *ex-ante* optimal mechanism is a posted price at $p = \frac{1}{2}$, which we can also easily deduce directly: there is no reason to randomize over suboptimal prices.

When $\gamma < 1$ the *ex-ante* optimal mechanism is not a posted price. To see this, compute the necessary first-order condition of (5),

$$\nabla . \mathcal{H}_{\nabla \nu} = \mathcal{H}_\nu,$$

where $\mathcal{H} = \int_0^1 \int_{v_s}^1 \left[\left(-2\frac{\nu(v_s,v_b)}{v_b - v_s} + \frac{\partial \nu}{\partial v_b}\right)^\gamma + \left(2\frac{\nu(v_s,v_b)}{v_b - v_s} + \frac{\partial \nu}{\partial v_s}\right)^\gamma\right] dv_b dv_s$, and ∇ is the gradient operator.[12] The expression for the first-order condition is somewhat messy, but it is immediate that, when $\gamma < 1$, a constant function does not solve this equation, so that no posted price is a solution when $\gamma < 1$. Since the *ex-ante* optimal simple mechanism is not a posted price it then follows that *ex-ante* optimal mechanism must be probabilistic. It is also clear that a lottery over posted prices cannot be optimal since when $\gamma < 1$ simple mechanisms dominate such lotteries by the representation of Proposition 13. Thus an *ex-ante* optimal mechanism under risk aversion necessarily has the feature that prices depend on traders valuations, so that if there is dispersion of valuations we should also observe dispersion in prices.

The analysis of symmetric CRRA environments is simple because we have the closed-form solutions for all simple *cexpiric* mechanisms. A similar exercise could be performed more generally, for set ups with $u_i(x) = x^{\gamma_i}$, $\gamma_s \neq \gamma_b$, but the computations would be numerical at all steps of the analysis.

6 Conclusion

We focused on the simplest exchange with a two-sided incomplete information. The key to our analysis is the use of the distribution-free concept of *ex-post* constrained efficiency, in conjunction with Proposition 7. These methods apply more generally. Immediate is the extension to the problem of providing a public good with private valuations, analogous to Mailath and Postlewaite [1990], but incorporating robustness and risk aversion.

The present results provide lower bounds for efficiency of optimal Bayesian mechanisms. We remark, however, that the efficiency results for *ex-post* implementation that we provide here hold for environments with correlated types. In contrast, under *interim* incentive and participation constraints, with correlation, full efficiency is possible (see Cremer and Maclean [1985,1988] and Mcafee and Reny [1992]).

In the present paper, we addressed the case where the mechanism designer knows the shape of the traders utilities. This information is necessary for the designer to know, in order to be able to construct incentive-compatible direct-revelation mechanisms. In Čopič and Ponsatí [2006], we show that when the mechanism designer does not know the shape of the traders' utility functions, but this information is known to the traders, the mechanism designer can construct an optimal indirect game form, the *mediated bargaining game*. The equilibria of the mediated bargaining game implement the *ex-post* constrained-efficient allocations described here.

[12] Here, $\mathcal{H}_{\nabla\nu}$ denotes the vector of partial derivatives of $\mathcal{H}$ w.r.t. all the components of $\nabla\nu$, and $\nabla.\mathcal{H}_{\nabla\nu}$ is the dot product of gradient operator with $\mathcal{H}_{\nabla\nu}$ - i.e., it is the sum of components of $\mathcal{H}_{\nabla\nu}$, each differentiated w.r.t. the appropriate component of v.

Appendix A

Proof of Lemma 2

Proof: We provide the proof for the seller. Fix a $v_b \in [0,1]$, and let $v_s, \bar{v}_s \in [0,1], v_s < \bar{v}_s$. By XPIC, we have

$$U_s^m(v_s, v_b) \geq U_s^m(v_s, v_b, \bar{v}_s), \text{ and}$$

$$U_s^m(\bar{v}_s, v_b) \geq U_s^m(\bar{v}_s, v_b, v_s).$$

We subtract these inequalities to obtain

$$E_{\mu[v_b,v_s]}\left[u_s(p-v_s) - u_s(p-\bar{v}_s)\right] \geq U_s^m(v_s, v_b) - U_s^m(\bar{v}_s, v_b),$$

$$U_s^m(v_s, v_b) - U_s^m(\bar{v}_s, v_b) \geq E_{\mu[v_b,\bar{v}_s]}\left[u_s(p-v_s) - u_s(p-\bar{v}_s)\right].$$

If $\bar{v}_s$ is close enough to v_s, then $u_s(p-\bar{v}_s) > -\infty$, for all $p \in \mathbf{supp}(\mu[v_s, v_b])$, since $u_s'(0) < \infty$, so that by continuity of u_s', $\exists \epsilon > 0$, s.t. $u_s'(x) < \infty$ for all $x \in [-\epsilon, 0]$. Thus, for $\bar{v}_s - v_s < \epsilon$, and $p \in \mathbf{supp}(\mu[v_s, v_b])$, $u_s(p-v_s) - u_s(p-\bar{v}_s) \leq u_s'(-\epsilon)(\bar{v}_s - v_s)$, so that

$$E_{\mu[v_b,v_s]}\left[u_s(p-v_s) - u_s(p-\bar{v}_s)\right] \leq E_{\mu[v_b,v_s]}\left[u_s'(-\epsilon)(\bar{v}_s - v_s)\right] \leq u_s'(-\epsilon)(\bar{v}_s - v_s).$$

To summarize, for $\bar{v}_s - v_s < \epsilon$,

$$U_s^m(v_s, v_b) - U_s^m(\bar{v}_s, v_b) \leq u_s'(-\epsilon)(\bar{v}_s - v_s),$$

proving that $U_s^m(v_s, v_b)$ is Lipschitz in v_s. ■

Proof of Proposition 7.

We will need the following Lemma to show that a nonsimple mechanism cannot *ex-post* dominate a simple one.

Lemma 15. Assume utilities depend only on the net surplus, $u_s(v_s, p) = u_s(p - v_s)$ and $u_b(v_b, p) = u_b(v_b - p)$, $u_i : [0,1] \to R, i = s, b$. Also assume that $u_i''(y) < 0, \forall y \in [0,1]$, for at least one i, and $u_i'''(y) \leq 0, \forall y \in [0,1], i = s, b$. Next, let μ be a measure with *supp*$(\mu) \subset [0,1]$, let

$$U_s^\mu = E_\mu[u_s(y)],$$

$$U_b^\mu = E_\mu[u_b(y)],$$

and define $p, \sigma \in [0,1]$ by $\sigma u_s(p) = U_s^\mu, \sigma u_b(1-p) = U_b^\sigma$. Then at least one of the following must be true:

1. μ is a degenerate point-mass at p and $\sigma = \mu[\{p\}]$,
2. $\sigma u'_s(p) < E_\mu[u'_s(y)]$, or
3. $\sigma u'_b(1-p) < E_\mu[u'_b(1-y)]$.

Proof: Suppose μ is nondegenerate. First note that p and σ are uniquely defined. Next, we can assume without loss of generality that by normalization, $\mu([0,1]) = 1$. Since $u''_i(y) < 0$, it follows by Jensen's inequality that

$$u_s(E_\mu y) \geq E_\mu[u_s(y)] = \sigma u_s(p),$$

$$u_b(E_\mu y) \geq E_\mu[u_b(y)] = \sigma u_b(1-p),$$

where at least one of the inequalities is strict, and $\sigma < 1$. If $E_\mu y \leq p$, then by convexity of u'_s and Jensen's inequality,

$$E_\mu u'_s(y) \geq u'_s(E_\mu y) \geq u'_s(p) > \sigma u'_s(p).$$

Alternatively, if $E_\mu y \geq p$, then by convexity of u'_b,

$$E_\mu[u'_b(1-y)] \geq u'_b(1 - E_\mu y) \geq u'_b(1-p) > \sigma u'_b(1-p).$$

■

Now we are ready to prove Proposition 7.

Proof: By XPIRIC, $U_i^m(v)$ is continuous and monotonic w.r.t. v_i, at each $v \in [0,1]^2$, s.t. $U_i^m(v) > 0$, implying that the left and the right limit of the partial derivative of $U_i^m(v)$ w.r.t. v_i exist. Thus, the XPIC constraints can be written as:

$$\frac{\partial^+ U_s^m(v)}{\partial v_s} \leq -E_{\mu[v]}[u'_s(x - v_s)] \leq \frac{\partial^- U_s^m(v)}{\partial v_s} \leq 0,$$

$$\frac{\partial^- U_b^m(v)}{\partial v_b} \geq E_{\mu[v]}[u'_b(v_b - x)] \geq \frac{\partial^+ U_b^m(v)}{\partial v_b} \geq 0.$$

This is easily verified using standard arguments. A mechanisms $m = \{\mu[v] \mid v \in [0,1]^2\}$ is differentiable at $v \in [0,1]^2$ if and only if the incentive constraints hold at v with equalities, i.e.,

$$\frac{\partial U_s^m(v)}{\partial v_s} = -E_{\mu[v]}[u'_s(x - v_s)],$$

$$\frac{\partial U_b^m(v)}{\partial v_b} = E_{\mu[v]}[u'_b(v_b - x)].$$

By regularity, $U_i^m(v)$ is absolutely continuous. Hence, for each v_j, $\frac{\partial U_i^m(v)}{\partial v_i}$ exists almost everywhere, and $U_i^m(v)$ is the integral of its derivative w.r.t. v_i. By XPIRIC, this gives

$$\begin{array}{l} U_s^m(v_s, v_b) = \int_{v_s}^{v_b} E_{\mu[\tau, v_b]}\left[u_s'(x-\tau)\right] d\tau, \\ U_b^m(v_s, v_b) = \int_{v_s}^{v_b} E_{\mu[\tau, v_s]}\left[u_b'(x-\tau)\right] d\tau. \end{array} \tag{6}$$

Now let m be simple, $m = (\varphi, \pi)$, and the allocation at $v = (0, 1)$ be Pareto optimal. Assume there exists an $\tilde{m}$ which *ex-post* dominates m. Assume first that $\tilde{m}$ is simple, $\tilde{m} = (\tilde{\varphi}, \tilde{\pi})$.

At $v = (0, 1)$ the allocation assigned by $\tilde{m}$ must be the same as the allocation under m, by Pareto optimality. Take the line $L_s(1) = \{(v_s, 1) \mid v_s \in [0, 1]\}$. By assumption, $U_s^m(v) \le U_s^{\tilde{m}}(v), \forall v \in L_s(1)$, and since the seller's XPIC constraints for m and $\tilde{m}$ hold almost everywhere on $L_s(1)$ with equality, we have by representation (6), that $U_s^m(v) = U_s^{\tilde{m}}(v), \forall v \in L_s(1)$. Thus,

$$\frac{\partial U_s^m(v)}{\partial v_s} = \frac{\partial U_s^{\tilde{m}}(v)}{\partial v_s}, \forall v \in L_s(1).$$

Since $\frac{\partial U_s^m(v)}{\partial v_s} = -\varphi(v) u_s'(\pi(v) - v_s)$, we therefore have

$$\varphi(v) u_s(\pi(v) - v_s) = \tilde{\varphi}(v) u_s(\tilde{\pi}(v) - v_s)$$

and

$$\varphi(v) u_s'(\pi(v) - v_s) = \tilde{\varphi}(v) u_s'(\tilde{\pi}(v) - v_s),$$

almost everywhere on $L_s(1)$.

These imply that $\varphi(v) = \tilde{\varphi}(v), \pi(v) = \tilde{\pi}(v)$, almost everywhere $L_s(1)$, so that $U_b^m(v) = U_b^{\tilde{m}}(v)$, almost everywhere on $L_s(1)$.

Similarly, define $L_b(0) = \{(0, v_b) \mid v_b \in [0, 1]\}$, and by an analogous argument we obtain $U_b^m(v) = U_b^{\tilde{m}}(v), \forall v \in L_b(0)$ and $U_s^m(v) = U_s^{\tilde{m}}(v)$, almost everywhere on $L_b(0)$.

Now take for instance a $v = (0, v_b) \in L_b(0)$ s.t. $U_s^m(v) = U_s^{\tilde{m}}(v)$ and let $L_s(v_b) = \{(v_s, v_b) \mid v_s \in [0, 1]\}$. As before, we obtain $U_s^m(v) = U_s^{\tilde{m}}(v), \forall v \in L_s(v_b)$. Thus, the set where $U_s^m(v) \ne U_s^{\tilde{m}}(v)$ has Lebesgue measure 0 and hence cannot be open. Similarly, the set where $U_b^m(v) \ne U_b^{\tilde{m}}(v)$ has Lebesgue measure 0, so that $\tilde{m}$ cannot *ex-post* dominate m.

Assume then that $\tilde{m}$ is not simple. By Pareto optimality at $v = (0, 1)$, $\tilde{m}$ has to be simple at $(0, 1)$. Thus,

$$\bar{v} = \sup_{v_s} \inf_{v_b} \{v \mid \tilde{m} \text{ simple at } v\}$$

is well defined, and $\bar{v} \in [0, 1]$. Moreover, by (6), $U_i^m(\bar{v}) = U_i^{\tilde{m}}(\bar{v}), i = s, b$, so that Lemma 15 applies, and $\tilde{m}$ cannot *ex-post* dominate m. ■

Proof of sufficiency of Proposition 9.

Proof: Consider $U_s(v, v_s')$. We show that for all all $v_s' \neq v_s$ the derivative of $U_s(v, v_s')$ w.r.t. v_s' is decreasing whenever $U_s(v, v_s') > 0$ (deviations that give negative expected utility cannot be profitable). We consider $v_s' > v_s$ (the case $v_s' < v_s$ is analogous). Thus compute

$$\frac{\partial U_s(v, v_s')}{\partial v_s'} = \varphi(v_s', v_b)\frac{\partial u_s(\pi(v_s', v_b), v_s)}{\partial p}\frac{\partial \pi(v_s', v_b)}{\partial v_s'} + \frac{\partial \varphi(v_s', v_b)}{\partial v_s'} u_s(\pi(v_s', v_b), v_s).$$

From the first order condition we can express

$$\frac{\partial \pi(v_s', v_b)}{\partial v_s'} = -\frac{\frac{\partial \varphi(v_s', v_b)}{\partial v_s'} u_s(\pi(v_s', v_b), v_s')}{\varphi(v_s', v_b)\frac{\partial u_s(\pi(v_s', v_b), v_s')}{\partial p}}.$$

Substituting this into the previous expression we get

$$\frac{\partial U_s(v, v_s')}{\partial v_s'} = \frac{\partial \varphi(v_s', v_b)}{\partial v_s'}\left[u_s(\pi_s(v_s', v_b), v_s) - \frac{\frac{\partial u_s(\pi(v_s', v_b), v_s)}{\partial p} u_s(\pi(v_s', v_b), v_s')}{\frac{\partial u_s(\pi(v_s', v_b), v_s')}{\partial p}}\right].$$

Observe that $\frac{\partial u_s(\pi(v_s', v_b), v_s)}{\partial p} < \frac{\partial u_s(\pi(v_s', v_b), v_s')}{\partial p}$. Moreover by XPIC $\frac{\partial \varphi(v_s', v_b)}{\partial v_s'} < 0$. To see this, one can use the standard argument of writing down the XPIC constraints for two types of the seller and then expressing the derivative of φ as the limit of taking one of the two types toward the other. Thus, whenever $u_s(\pi(v_s', v_b), v_s) > 0$, $\frac{\partial U_s(v_s; v_s', v_b)}{\partial v_s'}$ is a decreasing function of v_s, implying that the local maximum of U_s is unique, and is also a global maximum. Similarly for U_b. ■

Appendix B

In the proof of Lemma 11 we apply the following simple Lemma a few times.

Lemma 16. Let a function $g : [0,1]^2 \to [0,1]$ have the property that

$$g(v_1, v_2) = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} g(\tau, v_2) d\tau, \forall v_1, v_2 \in [0,1], v_1 < v_2. \tag{7}$$

Then $g(v_1, v_2) = g(v_1', v_2), \forall v_1, v_1', v_2 \in [0,1], v_1, v_1' < v_2$.

Proof: It is enough to prove that if a function $\bar{g} : [0,1] \to [0,1]$ has the property that $\bar{g}(x) = \frac{1}{x} \int_0^x \bar{g}(t) dt$, then $\bar{g}(x)$ must be a constant. We show that $\bar{g}$ is continuous and differentiable on $(0,1]$ and that it's derivative is 0 on $(0,1]$. We first show that $\bar{g}$ is continuous on $(0,1]$. Take an $\bar{\epsilon} > 0$, and for $\epsilon > 0$ take $x, x' \in [\bar{\epsilon}, 1], |x - x'| < \epsilon$. Since $\bar{g}$ is nonnegative and bounded by 1 we have

$$|\bar{g}(x) - \bar{g}(x')| = \left| \frac{1}{xx'} \left(\int_0^{x'} (x' - x) \bar{g}(t) dt + \int_0^{x'} x' \bar{g}(t) \right) dt \right| \leq \frac{2\epsilon}{\bar{\epsilon}^2},$$

implying that $\bar{g}$ is continuous on $[\bar{\epsilon}, 1], \forall \bar{\epsilon} > 0$, so that it is continuous on $(0,1]$. Now observe that on $(0,1]$, $\bar{g}$ is a product of two continuously differentiable functions, hence it is continuously differentiable. Since

$$x\bar{g}(x) = \int_0^x g(t) dt,$$

we can take derivatives to obtain $x\bar{g}'(x) = 0, \forall x \in (0,1]$, so that $\bar{g}'(x) = 0, \forall x \in (0,1]$, and the claim follows. ■

Proof of Lemma 11.

Proof: We prove the Lemma in two main steps. The idea behind the proof is to define for each φ satisfying the conditions of the Lemma a linear functional Λ_φ from the set of L^1-integrable functions on $[0,1]$ into reals. Then we can use the Riesz representation theorem which says that every such functional is representable by an integral with respect to some measure on $[0,1]$. In step 1 we define such Λ_φ in a very intuitive and straightforward manner. This requires a standard measure-theoretic procedure via so-called *simple functions*. In step 2 we show that the functional Λ_φ from step 1 is well defined. This step requires some tedious algebra, which we divide into several substeps. In what follows, we assume that $\varphi : [0,1] \times [0,1] \to [0,1]$ is measurable and satisfies the XPIRIC, i.e., it satisfies the equation

$$\varphi(v_1, v_2) = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} (\varphi(v_1, v_2) + \varphi(\tau, v_2)) d\tau, \forall v_1, v_2 \in [0,1], v_1 < v_2.$$

In step 2, we will then impose additional conditions on φ, and slowly relax them in each substep as we proceed.

Step 1. Define the functional Λ_φ as follows. Let $0 \leq a < b \leq 1$ and let $1_{[a,b)}$ denote the indicator function of the interval $[a, b)$, i.e., $1_{[a,b)}(x) = 1$ if $x \in [a, b)$ and $1_{[a,b)}(x) = 0$ otherwise. Define

$$\Lambda_\varphi \left(1_{[a,b)}\right) = \varphi(a, b).$$

For a constant $\alpha \in R$, define $\Lambda_\varphi \left(\alpha 1_{[a,b)}\right) = \alpha\varphi(a, b)$. Note that we need to check that Λ_φ is well defined. In particular, it should be that if we take a point $c \in [a, b)$, then $\Lambda_\varphi \left(\alpha 1_{[a,b)}\right) = \Lambda_\varphi \left(\alpha 1_{[a,c)}\right) + \Lambda_\varphi \left(\alpha 1_{[c,b)}\right)$, since clearly $\alpha 1_{[a,c)} + \alpha 1_{[c,b)} = \alpha 1_{[a,b)}$. We do this in step 2.

Now we extend the definition of Λ_φ to the whole domain $L^1([0, 1])$ (the domain of Lebesgue integrable functions on $[0, 1]$). To do this, recall that a *simple function* in $L^1([0, 1])$ is defined as a function that takes only a finite number of values $\alpha_1, ..., \alpha_m$, for some finite m. (simple functions are used for instance when one defines the Lebesgue integral). Thus, such g_n can be written as a finite sum

$$g_n = \sum_{i=1}^{m} \alpha_i 1_{\Omega_i},$$

where each Ω_i is a measurable set, and $\cup_{i=1}^m \Omega_i = [0, 1]$. For a measurable set $\Omega_i \subset [0, 1]$, define $\Lambda_\varphi (1_{\Omega_i})$ in the obvious way, and for every simple function g_n, $\Lambda_\varphi (g_n)$ is then defined by linearity.

Take a function $g \in L^1([0, 1])$. Then there exists a monotone sequence of simple functions, $g_n \in L^1$, such that $g_n \to g$ in the L^1 norm. Finally, define

$$\Lambda_\varphi (g) = \lim_{n\to\infty} \Lambda_\varphi (g).$$

Thus, Λ_φ is formally defined, which concludes step 1.

Step 2. In this step we show that Λ_φ is well defined. It is enough to show that Λ_φ is well defined on the set of characteristic functions, as the rest follows by the monotone convergence theorem. Thus, it is enough to show, that $\varphi(a, b) = \varphi(a, c) + \varphi(c, b)$ for every triplet $0 \leq a \leq b \leq c \leq 1$. We break the proof into two cases. The first case is when φ which is continuous on $[0, 1]^2$ in each argument. The second case completes the proof for general φ.

Case 2.1. Let $\varphi(v_s, \tau)$ and $\varphi(\tau, v_b)$ be continuous in τ, for every $(v_s, v_b) \in [0, 1]^2$.

We define $\phi(v_s, v_b, t) = \varphi(v_s, t)+\varphi(t, v_b)-\varphi(v_s, v_b)$, and we prove that $\phi(v_s, v_b, t) = 0, \forall t \in [v_s, v_b]$. Note that ϕ is continuous in each of its arguments, in particular it is continuous in t. We proceed as follows. In step 2.1.1 we show that there exists a

$\bar{t} \in (v_s, v_b)$ s.t. $\phi(v_s, v_b, \bar{t}) = 0$. In step 2.1.2 we show that $\frac{\partial \phi(v_s,v_b,t)}{\partial t} = 0$ everywhere by showing that the derivative of $\phi(v_s, v_b, t)$ w.r.t. t from the left is equal to that derivative from the right everywhere (and both are equal to 0). From the definition of ϕ it is clear that its derivative from the left w.r.t. t will be equal to 0 if and only if the derivative from the left of $f(v_s, t)$ w.r.t. t is equal the derivative from the left of $f(t, v_b)$ w.r.t. t, which is precisely what we show in step 2.1.2. Similarly for the derivative from the right. Thus, ϕ is differentiable, its derivative is 0, and it is equal to 0 at some point by step 1.1 - then ϕ must be equal to 0 everywhere. While step 1.1 is straightforward, step 2.1.2 involves some calculus.

Step 2.1.1. There exists a $\bar{t} \in (v_s, v_b)$ s.t. $\phi(v_s, v_b, \bar{t}) = 0$.

Proof. Now (2) can be written as

$$0 = \frac{1}{v_b - v_s} \int_{v_s}^{v_b} \phi(v_s, v_b, \tau) d\tau.$$

By the mean value theorem (MVT), there exists a $\bar{t} \in (v_s, v_b)$, s.t. $\frac{1}{v_b - v_s} \int_{v_s}^{v_b} \phi(v_s, v_b, \tau) d\tau = \phi(v_s, v_b, \bar{t})$, which concludes the proof of step 2.1.1.

Step 2.1.2. $\phi(v_s, v_b, t)$ is differentiable in t and $\frac{\partial \phi(v_s,v_b,t)}{\partial t} = 0$, for all $t \in (v_s, v_b)$.

Proof. Denote by

$$\frac{\partial^+ \varphi(v_s, t)}{\partial t} = \lim_{\epsilon \to 0,\ \epsilon > 0} \frac{\varphi(v_s, t + \epsilon) - \varphi(v_s, t)}{\epsilon}$$

the derivative from the right of $\varphi(v_s, t)$ w.r.t.t. Similarly, let $\frac{\partial^- \varphi(v_s,t)}{\partial t}$ denote the derivative from the left. We will show that for every $t \in (v_s, v_b)$,

$$\frac{\partial^+ \phi(v_s, v_b, t)}{\partial t} = \frac{\partial^- \phi(v_s, v_b, t)}{\partial t} = 0.$$

We will do that by showing that $\frac{\partial^+ \varphi(v_s,t)}{\partial t} = -\frac{\partial^+ \varphi(t,v_b)}{\partial t}$ and $\frac{\partial^- \varphi(v_s,t)}{\partial t} = -\frac{\partial^- \varphi(t,v_b)}{\partial t}$, for all $t \in (v_s, v_b)$. Note that the left and the right-derivatives of $\varphi(v_s, t)$ and $\varphi(t, v_b)$ w.r.t.t exist for all t since φ is continuous and monotonic.

We first show that

$$\frac{\partial^+ \varphi(v_s, t)}{\partial t} = \frac{\partial \varphi^+(v_s', t)}{\partial t}, \forall v_s', v_s < t. \tag{8}$$

To see this, we write by definition,

$$\frac{\partial^+ \varphi(v_s,t)}{\partial t} = \lim_{\epsilon\to 0,\ \epsilon>0} \frac{1}{\epsilon}\left(\varphi(v_s,t+\epsilon)-\varphi(v_1,t)\right).$$

We now use (2) and compute

$$\varphi(v_s,t+\epsilon)-\varphi(v_s,t) = \int_{v_s}^{t+\epsilon} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} d\tau - \int_{v_s}^{t} \frac{\varphi(v_s,\tau)+\varphi(\tau,t)}{t-v_s} d\tau$$
$$= \int_{t}^{t+\epsilon} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} d\tau + \int_{v_s}^{t} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} - \frac{\varphi(v_s,\tau)+\varphi(\tau,t)}{t-v_s} d\tau$$
$$= \int_{t}^{t+\epsilon} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} d\tau + \int_{v_s}^{t} \frac{-\epsilon(\varphi(v_s,\tau)+\varphi(\tau,t))}{(t+\epsilon-v_s)(t-v_s)} + \frac{\varphi(\tau,t+\epsilon)-\varphi(\tau,t)}{t+\epsilon-v_s} d\tau$$
$$= \int_{t}^{t+\epsilon} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} d\tau - \frac{\epsilon\varphi(v_s,t)}{t+\epsilon-v_s} + \int_{v_s}^{t} \frac{\varphi(\tau,t+\epsilon)-\varphi(\tau,t)}{t+\epsilon-v_s} d\tau.$$

From this last expression we can see that $\lim_{\epsilon\to 0,\epsilon>0} \frac{1}{\epsilon}\left(\varphi(v_s,t+\epsilon)-\varphi(v_s,t)\right) = \frac{1}{t+\epsilon-v_s}\int_{v_s}^{t} \frac{\partial^+\varphi(\tau,t)}{\partial v_b} d\tau$, since

$$\lim_{\epsilon\to 0,\epsilon>0} \frac{1}{\epsilon}\int_{t}^{t+\epsilon} \frac{\varphi(v_s,\tau)+\varphi(\tau,t+\epsilon)}{t+\epsilon-v_s} d\tau - \frac{\varphi(v_s,t)}{t+\epsilon-v_s} = 0,$$

by the MVT.

By Lemma 16 this implies that indeed (8) holds. Similarly, we obtain $\frac{\partial^+\varphi(t,v_b)}{\partial t} = \frac{\partial\varphi^+(t,v_b')}{\partial t}, \forall v_b', v_b > t.$

Now take a monotonic sequence $\epsilon_n, n = 1, ..., \infty$, s.t. $\lim_{n\to\infty}\epsilon_n = 0$, and let $v_{b,n}' = t+\epsilon_n$. By above, for each n,

$$\lim_{l\to\infty, l\geq n} \frac{\varphi(t+\epsilon_l, v_{b,n}')-\varphi(t,v_{b,n}')}{\epsilon_l} = \frac{\partial^+\varphi(t,v_{b,n}')}{\partial t} = \frac{\partial^+\varphi(t,v_b)}{\partial t}.$$

Then, by the Cauchy diagonalization theorem,

$$\lim_{n\to\infty} \frac{\varphi(t+\epsilon_n, v_{b,n}')-\varphi(t,v_{b,n}')}{\epsilon_n} = \frac{\partial^+\varphi(t,v_b)}{\partial t}. \tag{9}$$

Next, since $\varphi(t,t) = 0$, and also applying (8), we have for ϵ_n sufficiently small (i.e., n large enough),

$$\varphi(t,v_{b,n}') = \varphi(t,t+\epsilon_n) = \varphi(t,t) + \frac{\partial^+\varphi(t,t)}{\partial v_b}\epsilon_n + O(\epsilon^2) = \frac{\partial^+\varphi(v_s,t)}{\partial v_b}\epsilon_n + O(\epsilon_n^2).$$

Note that $\frac{\partial^+\varphi(t,t)}{\partial v_b}$ is understood as $\lim_{v_b\to t, v_b>t}\frac{\partial^+\varphi(t,v_b)}{\partial v_b}$. We insert this into (9), also noting that $\varphi(t+\epsilon_n, v'_{b,n}) = \varphi(t+\epsilon_n, t+\epsilon_n) = 0$, to obtain

$$\frac{\partial^+\varphi(t,v_b)}{\partial t} = \lim_{n\to\infty}\frac{\varphi(t+\epsilon_n, v'_{b,n}) - \varphi(t, v'_{b,n})}{\epsilon_n} = \lim_{n\to\infty}\frac{-\frac{\partial^+\varphi(v_s,t)}{\partial v_b}\epsilon_n + O(\epsilon_n^2)}{\epsilon_n} = -\frac{\partial^+\varphi(v_s,t)}{\partial v_b}.$$

Thus we have shown that at every $t\in(v_s,v_b)$, $\frac{\partial^+\varphi(t,v_b)}{\partial t} = -\frac{\partial^+\varphi(v_s,t)}{\partial v_b}$, which implies that $\frac{\partial^+\phi(v_s,v_b,t)}{\partial t}$ exists and is equal to 0. Similarly, we show that $\frac{\partial^-\phi(v_s,v_b,t)}{\partial t}$ exists and is equal to 0, which proves that $\phi(v_b,v_b,t)$ is differentiable w.r.t.t. This concludes the proof of step 2.1.2 and case 2.1.

Case 2.2. We complete the proof by showing that if $\varphi(v_s,v_b)$ is discontinuous things do not change, i.e., $\varphi(v_s,v_b)$ can only be discontinuous in a way which still preserves additive separability. In particular, we show that there exists a step function $\underline{\varphi}$: $[0,1]\times[0,1]\to[0,1]$ s.t. $\varphi(v_s,v_b)-\underline{\varphi}(v_s,v_b)$ is continuous, and $\underline{\varphi}(v_s,v_b) = \underline{\tilde{\varphi}}(v_b) - \underline{\tilde{\varphi}}(v_s)$, for some step function $\underline{\tilde{\varphi}}:[0,1]\to[0,1]$. We proceed in 2 steps, both involve applying the Monotone Convergence Theorem (MCT), and some tedious calculus.

Step 2.2.1. If $\exists v_s\in[0,1]$, and $\bar\tau > v_s$ s.t. $\varphi(v_s,\bar\tau+) - \varphi(v_s,\bar\tau-) = \Delta_s(v_s,\bar\tau) > 0$, then $\varphi(v'_b,\bar\tau+) - \varphi(v'_s,\bar\tau-) = \Delta_s(v_s,\bar\tau) > 0$, $\forall v'_s < \bar\tau$.

Proof. We write

$$\varphi(v_s,\bar\tau+) = \lim_{\epsilon\to 0}\frac{1}{\bar\tau+\epsilon-v_s}\int_{v_s}^{\bar\tau+\epsilon}\varphi(v_s,\tau)+\varphi(\tau,\bar\tau+\epsilon)d\tau,$$

$$\varphi(v_s,\bar\tau-) = \lim_{\epsilon\to 0}\frac{1}{\bar\tau-\epsilon-v_s}\int_{v_s}^{\bar\tau-\epsilon}\varphi(v_s,\tau)+\varphi(\tau,\bar\tau-\epsilon)d\tau,$$

and since

$$\lim_{\epsilon\to 0}\frac{1}{\bar\tau+\epsilon-v_s} = \lim_{\epsilon\to 0}\frac{1}{\bar\tau-\epsilon-v_s} = \frac{1}{\bar\tau-v_s},$$

we have

$$\Delta_s(v_s,\bar\tau) = \frac{1}{\bar\tau-v_s}\left[\lim_{\epsilon\to 0}\int_{\bar\tau-\epsilon}^{\bar\tau+\epsilon}\varphi(v_s,\tau)d\tau + \lim_{\epsilon\to 0}\int_{v_s}^{\bar\tau+\epsilon}\varphi(\tau,\bar\tau+\epsilon)d\tau - \int_{v_s}^{\bar\tau-\epsilon}\varphi(\tau,\bar\tau-\epsilon)d\tau\right]. \tag{10}$$

Now

$$\lim_{\epsilon\to 0}\int_{\bar\tau-\epsilon}^{\bar\tau+\epsilon}\varphi(v_s,\tau)d\tau = \lim_{\epsilon\to 0}\int_{v_s}^{1} 1_{(\bar\tau-\epsilon,\bar\tau+\epsilon)}\varphi(v_s,\tau)d\tau = 0,$$

by the (MCT). Similarly, we apply the (MCT) to the other part of (10), so that

$$\lim_{\epsilon\to 0}\int_{v_s}^{\bar\tau+\epsilon}\varphi(\tau,\bar\tau+\epsilon)d\tau-\int_{v_s}^{\bar\tau-\epsilon}\varphi(\tau,\bar\tau-\epsilon)d\tau=\lim_{\epsilon\to 0}\int_{v_s}^{1}1_{(v_s,\bar\tau+\epsilon)}\varphi(\tau,\bar\tau+\epsilon)-1_{(v_s,\bar\tau+\epsilon)}\varphi(\tau,\bar\tau+\epsilon)d\tau$$

$$=\int_{[v_s,\bar\tau)}\varphi(\tau,\bar\tau+)-\varphi(\tau,\bar\tau-)d\tau.$$

Therefore,

$$\Delta_s(v_s,\bar\tau)=\frac{1}{\bar\tau-v_s}\int_{[v_s,\bar\tau)}\varphi(\tau,\bar\tau+)-\varphi(\tau,\bar\tau-)d\tau=\frac{1}{\bar\tau-v_s}\int_{[v_s,\bar\tau)}\Delta_s(\tau,\bar\tau)d\tau. \quad (11)$$

The claim now follows for $v_s<\bar v_s<\bar\tau$, by Lemma 16. This concludes the proof of step 2.2.1.

Step 2.2.2. If $\exists v_s\in[0,1]$, and $\bar\tau>v_s$ s.t. $\varphi(v_s,\bar\tau+)-\varphi(v_s,\bar\tau-)=\Delta>0$, then $\exists v_b>\bar\tau$ s.t. $\varphi(\bar\tau-,v_b)-\varphi(\bar\tau+,v_b)=\Delta$.

Proof. Since $\varphi(0,\tau)$ is bounded and monotonic, there exists a $\bar v_b$ s.t. $\varphi(0,\tau)$ is continuous for $\tau\in(\bar\tau,\bar v_b]$. By step 2.2.1, $\varphi(v_s,\tau)$ is continuous for $\tau\in(\bar\tau,\bar v_2],\forall v_s<\bar v_b$. We can proceed as in step 2.2.1 to obtain for each v_b,

$$\Delta_b(v_b,\bar\tau)=\frac{1}{v_b-\bar\tau}\left[\lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{v_b}\varphi(\bar\tau-\epsilon,\tau)d\tau-\int_{\tau+\epsilon}^{v_b}\varphi(\bar\tau+\epsilon,\tau)d\tau\right].$$

Next,

$$\lim_{\epsilon\to 0}\int_{\tau-\epsilon}^{v_b}\varphi(\bar\tau-\epsilon,\tau)d\tau-\int_{\tau+\epsilon}^{v_b}\varphi(\bar\tau+\epsilon,\tau)d\tau=\lim_{\epsilon\to 0}\int_{\tau+\epsilon}^{v_b}\varphi(\bar\tau-\epsilon,\tau)-\varphi(\bar\tau+\epsilon,\tau)d\tau+\int_{\tau-\epsilon}^{\tau+\epsilon}\varphi(\bar\tau-\epsilon,\tau)d\tau$$

$$=\lim_{\epsilon\to 0}\int_{\tau+\epsilon}^{v_b}\varphi(\bar\tau-\epsilon,\tau)-\varphi(\bar\tau+\epsilon,\tau)d\tau=\int_{(\tau,v_b]}\lim_{\epsilon\to 0}\varphi(\bar\tau-\epsilon,\tau)-\varphi(\bar\tau+\epsilon,\tau)d\tau,$$

where the second equality follows by MCT, and the third one by the bounded convergence theorem. Thus, for every v_b,

$$\Delta_b(v_b,\bar\tau)=\frac{1}{v_b-\bar\tau}\int_{(\tau,v_b]}\lim_{\epsilon\to 0}\varphi(\bar\tau-\epsilon,\tau)-\varphi(\bar\tau+\epsilon,\tau).$$

For each $k=1,...,\infty$, by continuity and monotonicity of $\varphi(\bar\tau+\frac{1}{k},\tau)$, and since $\varphi(\bar\tau+\frac{1}{k},\bar\tau+\frac{1}{k})=0$, there exists a $v_b^{(k)}>\bar\tau+\frac{1}{k}$, s.t. $\varphi(\bar\tau+\frac{1}{k},v_b^{(k)})<\frac{1}{k}$. On the other hand, $\varphi(\bar{\tau} - \frac{1}{k}, v_b^{(k)}) \geq \Delta$, so that

$$\Delta_b(v_b^{(k)}, \bar{\tau}) > \Delta - \frac{1}{k},$$

which by step 2.2.1 implies that $\Delta_b(v_b, \bar{\tau}) \geq \Delta$. By a symmetric argument, it must be that $\Delta \geq \Delta_b(v_2, \bar{\tau})$. This concludes the proof of step 2.2.2.

Now we wrap up the proof of the Lemma. Define $\underline{\tilde{\varphi}}(x)$ by the Lebesgue integral

$$\underline{\tilde{\varphi}}(x) = \int_0^x \Delta_b(0, y)dy.$$

By steps 2.2.1 and 2.2.2, $\varphi(v_s, v_b) - \left(\underline{\tilde{\varphi}}(v_b) - \underline{\tilde{\varphi}}(v_s)\right)$ is continuous, and we apply case 2.1 to conclude the proof of step 2, and thus the proof. ■

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