Robust Efficient Decision Rules

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Abstract

Robust efficient decision rules are not (*ex post*) Pareto dominated within the set of decision rules that satisfy some robust notion of incentive compatibility and other relevant constraints pertaining to the economic environment. Examples in public and private good environments show non-existence of classically (Pareto) efficient decision rules, followed by examples of anonymous robust efficient decision rules. Robust efficient decision rules are characterized as *ex ante* or *interim* incentive efficient decision rules in a robust framework.

1 Introduction

The discussion of efficiency is central to economics. When all the information in the economy is known by all individuals, Pareto efficiency – henceforth *classical efficiency* – has been the unassailable fundamental definition of an efficient allocation. Such classical conditions of complete information obtain when various pieces of relevant information, initially held by different individuals, have all been aggregated through some communication system, henceforth *decision rule*. For example, in an exchange economy if the individuals can credibly communicate their demands, market clearing prices may aggregate all relevant information

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and a classically efficient allocation may obtain.[1] In general, however, Arrow's impossibility theorem, and especially the closely-related Gibbard and Satterthwaite theorem have taught us that in some environments of incomplete information it is impossible to simultaneously satisfy classical efficiency, incentive compatibility, and some other desirable conditions.[2] Harsanyi's seminal description of Bayesian environments with incomplete information where the individuals may hold some (possibly subjective) prior beliefs over the uncertain parameters, [26], called for a definition of efficiency *prior* to when any communication of information has taken place or the basic decisions regarding production and allocation have been made. Wilson, [51], provided such a definition without considering the incentive constraints, and Hölmstrom and Myerson, [27], in yet another seminal contribution provided a definition of incentive efficiency for Bayesian environments with incomplete information.

Recently, there has been a lively discussion of environments where the allocation rules and the individuals' incentives are immune to the details of the specification of the individuals' information.[3] While robust notions of incentive compatibility are relatively well understood, there has been no similarly appealing general definition of robust incentive efficiency. Indeed, most of the literature concerning efficiency in such environments has focused on characterizing conditions under which robust classically efficient decision rules exist.[4] In their work, Hölmstrom and Myerson, [27] did define the notion of *ex post* incentive

[1]This fact is known as the First Welfare Theorem, which holds under certain conditions, in particular, when the number of individuals is large and there are no consumption externalities, see for example [34] and [41]. In exchange economies [42] showed that the Walrasian correspondence is not Bayesian incentive compatible with a finite number of individuals and [23] constructed Bayesian incentive compatible decision rules that converge to the Walrasian allocation as the number of individuals becomes large. Under the requirement of dominant strategy incentive compatibility, or *strategy proofness*, [28] showed that the Walrasian allocation is manipulable with a finite number of individuals and [8] showed that the Walrasian allocation is not strategy-proof even when the number of individuals becomes large.

[2]Gibbard and Satterthwaite, see [20] and [46], proved that under the full domain assumption, where each individual can hold any preference over the set of possible alternatives, any aggregation rule that is *strategy-proof* and classically efficient must be dictatorial so that there exists no strategy-proof and classically efficient aggregation rule that is responsive to the preferences of every individual. While Arrow, [2], does not explicitly describe the environment as one of incomplete information Philip Reny in his elegant exposition, [44], shows the intimate relationship between Arrow's theorem, and the Gibbard and Satterthwaite theorem.

[3]See [30] for an early work on incentive compatibility; [35] and [11] provided a formal definition of rich type spaces, and [9] embedded this into the mechanism design framework to study various robust notions of incentive compatibility, in particular, *ex post* incentive compatibility and *strategy proofness*.

[4]See especially [5] and [6], who characterize conditions under which strategy-proofness implies coalitional-strategy-proof decision rules; strategy-proofness with respect to the grand coalition implies classical efficiency. An exception is the work by Smith, [48], who in the context of an all-or-nothing public good decision, where classically efficient decision rules do not exist, defines a ranking based on existence of *some* belief under

efficiency, which is defined as constrained Pareto efficiency under the feasibility constraint of Bayesian incentive compatibility.[5] However, they viewed the *ex post* stage as one where all the information has become publicly available rather than *a description of informational constraints specific to robustness* as suggested by [9]. Hölmstrom and Myerson thus discarded the *ex post* incentive efficiency as a relevant notion of efficiency.

Our central assertion is that *ex post* constrained efficiency is precisely the right robust efficiency criterion. The feasibility constraints are given by some adequate robust notion of incentive compatibility and perhaps some other constraints relevant to the economic environment at hand. We thus define *robust efficient* decision rules as those which are not Pareto dominated within the set of feasible decision rules. This definition is elementary and whenever the feasibility constraints are not binding robust efficiency coincides with classical efficiency as it properly ought to. Following the approach of [9], robust efficient decision rules can also be thought of as those that are *ex ante* or *interim* incentive efficient under the individuals' sufficiently rich beliefs (Section 5). Our examples illustrate not only the usefulness of the notion as a conceptual tool, but also as an analytical tool (sections 3 and 4). In particular, in our second example, which is a special case of the aforementioned private-good environment with no consumption externalities, we provide a novel result of non-existence of classically efficient decisions rules and construct a decision rule that is robust efficient.[6] To facilitate these examples and to follow some of the discussion in Hölmstrom and Myerson on how and whether a robust efficient decision rule might be selected in a decentralized economy, we pay some special attention to anonymous decision rules (Section 6).[7]

which a decision rule is a strict improvement.

[5]A Bayesian incentive compatible decision rule is *ex post* incentive efficient if there does not exist another Bayesian incentive compatible decision rule which Pareto dominates the former in every informational state.

[6]Further applications of robust efficiency may be found in [15], [16], [17], [1], [13] and [14].

[7]We use the definitions of symmetric environments and anonymous decision rules as well as the corresponding results of Section 6 in our earlier sections 3 and 4. However, these definitions are intuitive and we find the present organization much more consistent with the flow of the narrative so that we hope that the readers will excuse us the slight inconvenience on the strictly formal level.

2 Robust efficiency and simple implications

There are n individuals, $i \in N = \{1, ..., n\}$. For each individual i there is a finite set of payoff types, Θ_i, and $\mathbf{\Theta} = \Theta_1 \times ... \times \Theta_n$; denote by $\bar{\mathbf{\Theta}}$ the set of probability measures over $\mathbf{\Theta}$.[8] There is a set of (deterministic) feasible allocations Y, and $\bar{Y}$ is the set of randomized allocations. We will for the most part assume that Y is given by a finite number of alternatives $y_0 \in Y_0$ and the vectors of transfers y between the individuals, $y_N \in Y_1 \times ... \times Y_n$, $Y_i = \mathbb{R}$, where $y_i > 0$ when i receives a payment, and $y_i < 0$ when i makes a payment. An allocation $y \in Y$ is thus given by $y = (y_0, y_N)$.[9] Most of our definitions extend easily to the environments with no transfers, simply by setting $Y \equiv Y_0$.

Given a $\theta \in \mathbf{\Theta}$, individual i's preferences are given by a von Neumann-Morgenstern utility function $u_i(., \theta) : \bar{Y} \to \mathbb{R}$, where $u_i(\bar{y}, \theta)$ is linear in $\bar{y}$, since $\bar{y}$ is a probability measure. The payoff parameter θ_i incorporates all of i's payoff-relevant private information, θ_i is not known by other individuals, and the utility functions $u_i(., .)$ are common knowledge. In general, we assume that u_i is increasing in the transfer y_i, for each $y \in Y$, and each $\theta \in \mathbf{\Theta}$. In our examples we make the more restrictive assumption of quasi-linearity, that is, the utility functions are additively separable in the allocation and the monetary transfer, quasi-linear with respect to transfers, and are normalized so that 1 unit of money is worth 1 unit of utility to all agents, $u_i(y, \theta) = y_i + \nu_i(y_0, \theta)$.

For the purpose of this section it is common knowledge that the individuals' types lie in the parameter space $\mathbf{\Theta}$. The individuals have no further information regarding the details of the descriptive statistics of these types, in particular, they do not know the probability distribution over types.[10] In an environment with private values, where i's preferences are

[8] We use boldface notation $\mathbf{\Theta}$ to denote the entire parameter space to not cause confusion in symmetric environments, defined below, where we will use Θ to denote an individual's set of private parameters. For a set X we use $\bar{X}$ to denote the set of probability measures over X.

[9] For example, N may be a set of sellers and buyers of some number of objects, in which case Y is the set of possible allocations of the objects and the vector of transfers between the individuals. Different allocations may entail different production costs, and so on.

[10] In the social choice literature such a specification of an environment with incomplete information is standard; equilibrium (or other solution) concepts that are considered do not depend on the probability distribution over individuals' types, e.g., strategy-proofness. Alternatively, one can imagine a Bayesian approach to incomplete information started by [26], and then require that the model be immune to the changes of the structure of the environment – for example, robust to all possible individuals' priors as in [30].We refer to Section 5 for such a description in the framework of rich type spaces as in [9].

affected only by his own private parameter θ_i, the present description defines a (restricted) domain of i's preferences, $\{u_i(., \theta_i), \theta_i \in \Theta_i\}$. A decision rule (or a social choice function) is a mapping $d : \mathbf{\Theta} \to \bar{Y}$. We denote by $\mathcal{D}$ the set of all decision rules such that $\sum_{i \in N} d_i(\theta) \leq K$, $\forall \theta \in \mathbf{\Theta}$, for all $d \in \mathcal{D}$ and a constant $K < \infty$, in particular, we prohibit decision rules that require infinite subsidies.

Efficiency of decision rules is a predominant concern of classical economics. Central to that is the Pareto dominance relation.

Definition 1. *A decision rule d' Pareto (ex post) dominates d, denoted $d' \succ d$, iff*

$$u_i(d'(\theta), \theta) \geq u_i(d(\theta), \theta), \qquad \forall \theta \in \mathbf{\Theta}, \ \forall i \in N, \tag{1}$$

with at least one strict inequality. A decision rule $d \in \mathcal{D}$ is classically efficient *(or Pareto efficient or ex post efficient), if there does not exist a $d' \in \mathcal{D}$, such that d' Pareto dominates d.*

Pareto dominance relation depends on the parameter domain $\mathbf{\Theta}$ and the payoff information, but is independent of further details of the specification of individuals' information. Therefore, the notions of Pareto dominance relation and classical efficiency are both robust.

If in a given environment there are no constraints on the set of feasible decision rules, then classical efficiency is easily attainable. For example, in any quasi-linear environment with transfers and no constraints, a decision rule is efficient as long as $d_0(\theta) = \max_{y_0 \in Y_0} \sum_i \nu_i(y_0, \theta)$. However, in most any economic environment, the relevant decision rules must satisfy some constraints pertinent to the informational and contractual aspects of the allocation problem. Decision rules that do not satisfy the constraints are simply infeasible, for one reason or another. The set of feasible decision rules is given by some $D \subset \mathcal{D}$; classically efficient decision rules may or may not be feasible. Nevertheless, there may still be rules which are more efficient than others in the sense of Pareto domination. This suggests a natural way to extend the definition of classical efficiency to general robust environments, in particular, to those where classically efficient decision rules are infeasible. The following is the central notion of this study.

Definition 2. *Given a $D \subset \mathcal{D}$, a decision rule $d \in D$ is robust efficient in D if there does not exist a $d' \in D$, such that d' Pareto dominates d.*

We denote by $D^* \subset D$ the set of robust efficient decision rules.[11] Note that if there are no constraints, then the set $\mathcal{D}^*$ of robust efficient decision rules in $\mathcal{D}$ is the set of classically efficient decision rules.

A different approach to defining constrained efficient decision rules is by maximizing a social welfare criterion. When considering *ex post* constrained efficient decision rules, the appropriate measurability requirement is that the the weights in the social welfare function be measurable with respect to the individuals' types, see [27]. Given a $D \subset \mathcal{D}$, a decision rule $d \in D$ is a robust welfare maximizer if,

$$\begin{gathered} \exists \lambda : \mathbf{\Theta} \to R^N_{++}, \text{ s.t.}, \\ d \in \arg\max_{d' \in D} \textstyle\sum_{\theta \in \mathbf{\Theta}} \sum_{i \in N} \lambda_i(\theta) u_i(d(\theta), \theta). \end{gathered} \tag{2}$$

Denote by $D^{\mathcal{W}} \subset D$ the set of robust welfare maximizers. The next theorem is a straightforward extension of well-known standard results, see [51], and especially[27]. Denote the closure of a set O by $cl(O)$. Note that in a given decision rule the individuals' utilities are specified by $n \times |\mathbf{\Theta}|$ points in the Euclidean space. As far as Pareto dominance is concerned, we can limit attention to the space of the individuals' payoffs endowed with Euclidean metric and the corresponding topology.

Theorem 1. *Let $D \subset \mathcal{D}$ be closed and convex. Then $cl(D^{\mathcal{W}}) = D^*$.*

Most economic constraints are convex and closed. Before turning to our examples we enumerate some of these constraints, all of which are standard.

A key constraint is incentive compatibility. Under any specification such that the revelation principle holds, incentive compatibility requires that the decision rule be attainable as an equilibrium outcome mapping of some game form. More precisely, given an appropriate equilibrium notion, the revelation principle states that for any equilibrium outcome mapping of any game form there exists an incentive compatible decision rule $d(.)$ such that $d(\theta)$ is

[11] Such decision rules are indeed robust *constrained* efficient: the constraints are specified with D so that no confusion should arise with the notion of classical efficiency, i.e., unconstrained efficiency; we use the term robust efficient in D, meaning robust under the specific constraints describing D.

precisely the equilibrium outcome of that game form, for every realization of parameters $\theta \in \mathbf{\Theta}$.[12] The notion of incentive compatibility must correspond to the equilibrium notion, e.g., for dominant-strategy equilibrium the equivalent notion is the dominant-strategy incentive compatibility (or strategy-proofness). The contrapositive of the revelation principle is that if a decision rule d does not satisfy any notion of incentive compatibility then there does not exist any game form such that the decision rule d is attainable as an equilibrium mapping of that game form. In that case, d is simply not feasible as a representation of the workings of any institution. Therefore, some notion of incentive compatibility appropriate for the environment at hand is a minimal requirement that a feasible decision rule ought to satisfy.

Two standard notions of incentive compatibility suitable for robust environments are the dominant-strategy incentive compatibility and the somewhat weaker *ex post* incentive compatibility.[13]

Definition 3. *A decision rule d is dominant-strategy incentive compatible, if,*

$$u_i(d(\theta_i, \theta'_{-i}), \theta) \geq u_i(d(\theta'_i, \theta'_{-i}), \theta), \forall \theta_i, \theta'_i, \theta_{-i}, \theta'_{-i}.$$

Definition 4. *A decision rule d is (ex post) incentive compatible, if,*

$$u_i(d(\theta_i, \theta_{-i}), \theta) \geq u_i(d(\theta'_i, \theta_{-i}), \theta), \forall \theta_i, \theta'_i, \theta_{-i}.$$

From now on, when we say incentive compatible, we mean the weaker *ex post* incentive compatibility; however, in our two main examples below that will not play a role.[14] When a decision rule satisfies incentive compatibility it is called a direct revelation mechanism, or simply a mechanism.

Some other standard constraints that may be imposed on decision rules are individual rationality and budget balance (or no subsidies).

[12] See, e.g., [38] for more on revelation principle. Here we only consider partial implementation, that is, we do not require that truthful reporting is the unique equilibrium of the direct revelation mechanism.

[13] See especially [9] for a detailed discussion of incentive compatibility and implementability of decision rules, or social choice functions, in a robust setting. In turns, [52] considers a notion of rationalizability, which is also independent of the individuals' prior beliefs. As our main concern here is with efficiency, we will, as far as strategic concerns, limit our attention to *ex post* incentive compatibility.

[14] In a private-values environment – as is the case in our examples of Sections 3 and 4 – it is immediate that the above two notions of incentive compatibility are equivalent.

If each individual can under any circumstances obtain some base level of utility, usually normalized to 0, then any feasible mechanism must satisfy individual rationality, that is,[15]

$$u_i(d(\theta), \theta) \geq 0, \forall \theta, \forall i.$$

Budget balance is a reasonable requirement for environments where there is no agency capable of providing external subsidies to the system. That is, in every state θ, the sum of the transfers must be non-positive (or zero, in the case of exact budget balance) so that there need not be any external source of funds. A decision rule d satisfies (*ex post*) budget balance, if,

$$\sum_{i=1}^{n} d_i(\theta) \leq 0, \forall \theta,$$

and it satisfies exact budget balance if the above is an equality.[16]

In some more environments additional constraints may be required, for example, the *market clearing* constraints when Y_0 can be interpreted as a market – that is, the markets ought to clear for goods other than the numeraire.

The above constraints will define some subset of decision rules $D \subset \mathcal{D}$. These constraints – incentive compatibility, individual rationality, budget balance, or market clearing – assure that D is a closed and convex set so that Theorem 1 applies. An important example of non-convex constraints are those derived from voting correspondences, for example, when individual rationality constraints must be satisfied only by a simple majority of the individuals.

[15]For example, there may exist some $\tilde{y}^i \in Y_0$ such that when y is such that $y_0 = \tilde{y}^i$ and $y_{N,i} = 0$ then,

$$u_i(y, \theta) = 0, \forall \theta$$

In addition, for each i there may exist a type $\underline{\theta}_i$, which allows i to opt out, that is, such that when i reports that type, the social allocation is $\tilde{y}_i$. That is, there is a social allocation y such that when the transfer of individual i is 0, individual i obtains a zero utility.

[16]Recall that a decision rule is in general a mapping into lotteries over allocations. Therefore, individual rationality and budget balance as specified here may for each draw of types be interpreted either in terms of the expectation of that lottery, or as point-wise constraints for each realization of that lottery.

3 Example 1: Providing a public good

Our first illustrative example is a classical problem where N individuals must decide whether or not to finance a non-excludable, non-rival, indivisible indivisible public good. Thus, $Y_0 = \{0, 1\}$, where $y_0 = 1$ denotes the outcome whereby the public good is provided and $y_0 = 0$ the outcome whereby it is not provided. Each individual privately values the good at $\theta_i \in \Theta = \{v^L, v^H\}$, $0 < v^L < v^H < 1$, so that $\mathbf{\Theta} = \Theta^n$; the cost of the public good is normalized to 1. The individuals have quasilinear utilities, $u_i(y, \theta_i) = y_i + y_0\theta_i$, i.e., $\nu_i(y_0, \theta) = y_0\theta_i$. We further assume that,

$$(N-1)v^H + v^L > 1 \tag{3}$$

$$(N-2)v^H + 2v^L < 1 \tag{4}$$

In a classically efficient allocation the public good should be provided either if all individuals have a high valuation, since (3) implies that $Nv^H > 1$, or if $N-1$ individuals have a high valuation. Otherwise the good should not be provided by (4). When the individuals' reports of their valuations are θ, denote by $\varphi(\theta)$ the probability that the public good is provided. The transfer $y_i(\theta)$ is interpreted as the tax paid by individual i.

The set of feasible mechanisms is in principle given by direct revelation mechanisms requiring no external subsidies and where no individual can be forced to pay for the public good. We will impose an additional *no private benefits* constraint, or simply, no benefits. No benefits requires that no individual can directly benefit by receiving a monetary payment from the mechanism, that is, $y_i(\theta) \geq 0, \forall\theta, \forall i$. Every classically efficient decision rule ought to satisfy budget balance with equality. For that reason, and to avoid some complications, we further strengthen the budget balance requirement to exact budget balance.[17] The feasible set D is thus given by decision rules satisfying incentive compatibility, individual rationality, budget balance and no benefits.

It has been shown, under various circumstances, that no classically efficient allocation

[17]Some authors, e.g., [29], also argue that it is desirable to satisfy budget balance with equality for normative reasons. In settings with continuous type spaces and no budget breaker, [13] demonstrates that the standard representation of incentive compatible mechanisms is without loss of generality only with strict budget balance.

rules exist.[18] Consequently, a lively discussion has emerged along two different lines: to characterize decision rules satisfying certain additional assumptions, such as anonymity;[19] and to describe decision rules that are as efficient as possible in some set.[20] The present example illustrates this non-existence of classically efficient decision rule in the simplest possible public good environment and then characterizes the set of anonymous robust efficient decision rules. The characterization yields a clear description of efficiency losses associated with the above constraints.

Vickrey-Clarke-Groves (VCG) mechanisms.

A common class of mechanisms to consider are the Vickrey-Clarke-Groves (VCG) schemes (see [49], [12] and [22]). A VCG scheme is a demand revealing direct revelation mechanism which attains the allocation $y_0 \in Y_0$ that maximizes aggregate surplus. In this case, $\varphi \in \{0,1\}$, and $\varphi(\theta) = 1$, if and only if, $\sum_{i=1}^{N} \theta_i \geq 1$.

In particular, a VCG scheme with a fixed-size public good reduces to the *pivot* mechanism, see [21]. In the pivot mechanism, an individual i is pivotal if the allocation changes when she changes her report. We have two possibilities,

1. $(N-2)v^H + v^L - \frac{N-1}{N} \geq 0$.

[18]After initial impossibility results in ordinal environments in unrestricted domains by [2], [20], and [46], [28] provided an early impossibility theorem for incentive compatible, individually rational and Pareto efficient decision rules for private goods. [45] shows that when the number of individuals is very large, the Lindahl equilibrium allocation rule, which is classically efficient and individually rational is not even approximately incentive compatible. [50], [54], [7], and [47] showed non-existence of strategy-proof, efficient, and anonymous decision rules, and [41] provided a characterization by VCG mechanisms showing that no individual can impose a consumption externality on the others, which results in non-existence with any finite number of individuals. With [19] and [3] impossibility results a separate strain of literature addressed *interim* (Bayesian) incentive compatibility. [39] famously provided impossibility under the *interim* individual rationality, budget balance and incentive compatibility constraints in a two-person private good economy (or, equivalently, a two person public good economy), [24] studied expected welfare-maximizing public-good provision mechanisms and [33] showed that as the number of individuals tends to infinity, the probability of providing the public good tends to zero, and [32] characterize *interim* incentive efficient mechanisms.

[19][37] considers serial cost-sharing anonymous mechanisms for excludable public goods, [47] characterizes symmetric mechanisms for divisible public goods with convex costs as ascending-auction like mechanisms, [10] characterize symmetric coalition-proof mechanisms, and [29] characterize deterministic mechanisms.

[20][31] provides a review of public-good provision mechanisms and argues for comparisons on the grounds of efficiency of outcomes as well as transparency of the mechanisms themselves, [48] shows that as long as each individual might be willing to pay for the public good by herself, there are robust decision rules that under some beliefs do better than strategy-proof voting rules, building on [25], [17] characterize the robust efficient decision rules in the bilateral case of one private good, and under an additional condition of existence of a pivotal type [1] characterize general robust-efficient mechanisms satisfying the present assumptions in a continuous setting as lotteries over fixed-payments mechanisms.

Then,

$$y_i(\theta) = \begin{cases} \frac{1}{N}, & \text{if } \sum_{i=1}^{N} \theta_i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. $(N-2)v^H + v^L - \frac{N-1}{N} < 0$.
   Observe that $(N-3)v^H + 2v^L - \frac{N-1}{N} \geq 0$, by (3) and (3). Then, $y_i(\theta) = \frac{1}{N}$, if $\theta_i = v^H, \forall i$, and $y_i(\theta) = 0$, if there are at least 2 individuals with a low valuation, i.e., $\theta \in \mathbf{\Theta}$, s.t., $|\{\theta_i = v^L\}| \geq 2$. When θ is such that $|\{\theta_i = v^L\}| = 1$, then of course $\varphi(\theta) = 1$, and

$$y_i(\theta) = \begin{cases} \frac{1}{N}, & \text{if } \theta_i = v^L, \\ 1 - \left((N-2)v^H + v^L\right), & \text{if } \theta_i = v^H. \end{cases}$$

In both of these possibilities it is immediate to verify incentive compatibility. However, in the first possibility, the mechanism fails individual rationality of the individual with a low valuation when the public good is provided (since $v^L < \frac{1}{N}$, by (3) and (4)). In the second possibility, the mechanism does not fail individual rationality, however, when one individual has a low valuation the total payments exceed the cost of the public good. The reason is that $y_i(v^H) = 1 - \left((N-2)v^H + v^L\right) > \frac{1}{N}$, so that when θ is such that $|\{\theta_i = v^L\}| = 1$, we obtain,

$$\sum_{i=1}^{N} y_i(\theta) = y_i(v^L) + (N-1)y_i(v^H) = \frac{1}{N} + (N-1)y_i(v^H).$$

In the second possibility, while the allocation y_0 is efficient the mechanism is not classically efficient as some numeraire has to be disposed of. Therefore, in this environment, a VCG scheme is neither classically efficient nor does it belong to the feasible set D.[21]

We now consider the set of anonymous mechanisms in D; denote this set by $\bigcirc \subset D$. In an anonymous decision rule, when the names of individuals are permuted, so are their consumption bundles (see formal definitions 9, 10, and 11 in Section 6). Such a restriction

[21] One could of course think of an additional individual who absorbs the monetary surplus generated by the mechanism. That individual would have to be non-strategic and entirely disinterested in the public good. However, that seems at odds with the example of a non-rival public good. As soon as that additional individual has some incentives of her own, the aforementioned issues reemerge. An alternative way to state this is that the N individuals are considered as a closed system.

seems sensible given that all individuals are *a priori* identical so that it seems desirable that they be treated equally when determining the decision rule to be used. The set $\bigcirc$ is non-empty, for example, the mechanism where the good public good is never provided and no individual makes any payment satisfies all the requirements – this mechanism is evidently not particularly efficient and the question is whether there exist more efficient mechanisms in $\bigcirc$. We characterize the robust efficient mechanisms $\bigcirc^* \subset \bigcirc$.

Incentive compatibility will bind for the individuals to not misrepresent their values downward. Hence, all robust efficient mechanisms will have the property that the public good is provided with certainty when $\theta_i = v^H, \forall i$. Then, by symmetry, all individuals will pay the same tax, where the sum of these taxes is the cost of the good, so that $y_i = \frac{1}{N} < v^H$. Whenever more than one individual has a low valuation, individual rationality and budget balance imply that the public good is not provided and all taxes are 0.

The crucial case is when exactly one individual has a low valuation, that is $\theta \in \boldsymbol{\Theta}$ is s.t., $|\{i \mid \theta_i = v^L\}| = 1$. The good will then be provided with some probability, denoted by α. Conditional on the good being provided, by symmetry, all individuals with a high valuation will pay the same tax, denoted by y^H; denote by y^L the tax paid by the low valuation individual. We can therefore limit our attention to mechanisms specified by the triplet (α, y^H, y^L). For mechanisms in $\bigcirc$ these quantities satisfy the following constraints,

$$v^H - \frac{1}{N} \geq \alpha(v^H - y^L), \tag{5}$$

$$(N-1)y^H + y^L \geq 1, \tag{6}$$

$$v^H \geq y^H \geq 0, \tag{7}$$

$$v^L \geq y^L \geq 0. \tag{8}$$

(5) is the incentive constraint of high valuation individuals; for a low valuation individual, her incentive constraint is automatically satisfied since $v^L < \frac{1}{N}$. Equation (6) is the budget balance constraint, and (7) and (8) are the individual rationality and no benefits constraints of the high and the low valuation individuals, respectively.

For $\lambda \in [0,1]$, define,

$$\begin{aligned}\nu^H(\lambda) &= \lambda v^H + (1-\lambda)\frac{(1-v^L)}{N-1},\\ \nu^L(\lambda) &= \lambda(1-(N-1)v^H) + (1-\lambda)v^L,\\ \alpha(\lambda) &= \frac{v^H - \frac{1}{N}}{v^H - v^L + \lambda((N-1)v^H + v^L - 1)}\end{aligned}$$

For a $\lambda \in [0,1]$, define by d_λ the decision rule given by $(\alpha(\lambda), \nu^H(\lambda), \nu^L(\lambda))$. The next proposition characterizes the set of robust efficient public good provision mechanisms $\bigcirc^*$ in this environment.

Proposition 1. *The set $\bigcirc^*$ is given by the convex hull of the set $\{d_\lambda \mid \lambda \in [0,1]\}$.*

Proof. The key situation to consider is when all individuals except perhaps one have a high valuation of the public good. There are the two cases where the value of the good to that individual is either y^L or y^H. Given a y^L, the probability of providing the good must be maximal so that (5) must hold with equality, that is,

$$\alpha = \frac{v^H - \frac{1}{N}}{v^H - y^L}. \tag{9}$$

Similarly, (6) must hold with equality – otherwise taxes to some individuals can be reduced without reducing the probability of providing the good. Thus,

$$(N-1)y^H + y^L = 1.$$

In order to satisfy (7) and (8) the highest possible tax to the low valuation individual is v^L and the lowest is $1-(N-1)v^H$ so that admissible taxes to the low valuation individual are given by,

$$\nu^L(\lambda) = \lambda v^L + (1-\lambda)(1-(N-1)v^H), \lambda \in [0,1]$$

By exact budget balance admissible taxes to the high valuation individuals are given by $\nu^H(\lambda) = \frac{1-\nu^L(\lambda)}{N-1}$, and by substituting the expression for $\nu^L(\lambda)$ in place of y^L in (14) we obtain the expression for $\alpha(\lambda)$. By (3) and (4), $\alpha(\lambda) \in (0,1)$, for $\lambda \in (0,1)$ and by construction, $d_\lambda \in D$, $\forall \lambda \in [0,1]$.

To conclude the proof, we must first show that for $\lambda, \lambda' \in [0,1]$, d_λ does not dominate $d_{\lambda'}$; and second, that a mechanism given by $\mu d_\lambda + (1-\mu)d_{\lambda'}$ is undominated for $\mu \in [0,1]$.

For a given λ the utility of the low valuation individual is given by $\underline{U}(\lambda) = \alpha(\lambda)(v^L - \nu^L(\lambda))$, so that,

$$\underline{U}(\lambda) = (v^H - \frac{1}{N}) \times \frac{\lambda((N-1)v^H + v^L - 1)}{(v^H - v^L + \lambda((N-1)v^H + v^L - 1))},$$

which is increasing in λ, since the expression $\frac{\lambda((N-1)v^H+v^L-1)}{(v^H-v^L+\lambda((N-1)v^H+v^L-1))}$ is increasing in λ.

The utility of a high valuation individual is given by $\bar{U}(\lambda) = \alpha(\lambda)(v^H - \nu^H(\lambda))$, that is,

$$\bar{U}(\lambda) = \frac{(v^H - \frac{1}{N})}{(N-1)} \times \frac{\lambda((N-1)v^H + v^L - 1)}{(v^H - v^L + \lambda((N-1)v^H + v^L - 1))},$$

which is decreasing in λ, since $\frac{(1-\lambda)((N-1)v^H+v^L-1)}{(v^H-v^L+\lambda((N-1)v^H+v^L-1))}$ is decreasing in λ. Therefore, for $\lambda \neq \lambda'$, d_λ and $d_{\lambda'}$ do not dominate each other, one way or another.

Finally, it is easily shown that for $\lambda, \lambda' \in [0,1]$ and $\mu \in (0,1)$,

$$\mu\bar{U}(\lambda) + (1-\mu)\bar{U}(\lambda') > \bar{U}(\mu\lambda + (1-\mu)\lambda'),$$

while

$$\mu\underline{U}(\lambda) + (1-\mu)\underline{U}(\lambda') > \underline{U}(\mu\lambda + (1-\mu)\lambda').$$

Therefore, if the mechanism is given by $\mu d_\lambda + (1-\mu)d_{\lambda'}$, that is, d_λ with probability μ and $d_{\lambda'}$ with probability $1-\mu$, then such a randomized mechanism is also robust efficient. □

Note that the assumption of exact budget balance is non-essential in the present characterization – a similar characterization obtains under budget balance.

4 Example 2: Property rights

Our second example relates to several classical examples. In this second example there are N individuals who must allocate an object among themselves. *A priori* there are no property rights so that $Y_0 = N \cup \{0\}$, where $y_0 = i$ when the object is allocated to individual i and

$y_0 = 0$ when the object is not allocated. Each individual's valuation of the object is privately known and is given by $\theta_i \in \Theta$ for some finite set Θ. We again assume a quasilinear setting where $u_i(y, \theta) = y_i + 1_{\{y_0 = i\}}\theta_i$. We assume that no individual can be forced to participate in the scheme and that there is no outside subsidizing agency, so that the feasible set D is given by the decision rules satisfying incentive compatibility, individual rationality, and budget balance.

When it is common knowledge that the individual who values the object most knows that, this problem is known as "King Solomon's Dilemma." Then, there exist classically efficient allocations even when no transfers between the individuals are allowed (see [43], [40] and [4]). Another related problem is the partnership dissolution problem where the individuals initially hold property rights over shares of the object. Under the *interim* (Bayesian) constraints[18] provide conditions under which there exists a classically efficient solution.

When the object is allocated, transfers are made solely between the individuals. For classical efficiency these transfers must sum to zero – otherwise some numeraire would be disposed of. Denote by $\varphi_i(\theta)$ the probability that the object is allocated to i when reports are θ, and by $y_i(\theta)$ the transfer to/from the individual i. Proposition 2 states that in general there doesn't exist a classically efficient solution to this problem.

Proposition 2. *Let $N = 3$. Suppose that $\Theta_i = \Theta$, $\forall i$, $\{0, v^L, v^M, v^H\} \subset \Theta$. If $v^L < \frac{2}{3}v^M$ then $\nexists d \in D$, such that d is Pareto efficient.*

Proof. Suppose to the contrary, that there exists a classically efficient mechanism. By Proposition 4 it suffices to consider symmetric mechanisms. Note also that by classical efficiency budget balance implies exact budget balance.

Our proof now utilizes Table 1 below which describes probabilities φ_i and transfers y_i for different classes of type profiles θ. In the first row there are 10 different classes of profiles, up to permutations of individuals, and each class is denoted by a superscript. In the second row, if in a given class θ_i can take several different values then these values are described by the appropriate set, e.g., in the class θ^1, the valuation of individual 3, θ_3, can take any value in the set $\{v^M, v^L, 0\}$. A draw of types θ in the class θ^k is denoted by $\theta \in \theta^k$. Finally, α is the probability that the object is allocated to the individual with the second highest allocation when her valuation equals v^M and there are no ties; β denotes is that probability when her valuation equals v^L and there are not ties.

In a classically efficient mechanism, the object is allocated to an individual with the highest valuation of the good with probability 1, so that $\alpha = \beta = 0$. The following 5 steps demonstrate that the feasibility constraints pin down the allocation to that specified in Table 1. Consequently, no classically efficient mechanism exists in D.

Step 1. By symmetry and Pareto efficiency, at $\theta^0, \theta^4, \theta^{10}$, $\varphi_i = \frac{1}{3}$, and $y_i = 0, \forall i$. Additionally, set $\varphi_i(0,0,0) = 0, \forall i$, which is admissible as it is then efficient to dispose of the object. Note that setting $\varphi_i(0,0,0) = 0$ will yield highest efficiency as it provides the strongest possible incentives for higher types to not misrepresent downwards.

Step 2. At θ^1, $\varphi_1 = \varphi_2 = \frac{1}{2}$, $\varphi_3 = 0$, $y_1 = y_2 = -\frac{1}{6}v^M$, $y_3 = \frac{1}{3}v^M$.
Proof. Consider individual 3. When $\theta_1 = \theta_2 = v^H$, as long as $\theta_3 < v^H$, since $\varphi_3 = 0$, individual 3 must receive the same transfer regardless of her report as she would otherwise have incentives to misrepresent her valuation for those values of θ_3 where she obtains the smallest compensation. Therefore, the allocation is indeed constant at θ^1.

Next, individual 3 can be compensated at most $\frac{v^H}{3}$ when $\theta_3 = v^M$ in order for her not to misrepresent her valuation to v^M when $\theta = \theta^0$. On the other hand, at $\theta \in \theta^1$, for 3 not to misrepresent her valuation to v^H, she must be compensated at least $\frac{v^M}{3}$. Since the transfers must sum to 0, and by symmetry, we have $y_i(\theta^1) = -\frac{y_1(\theta^1)}{2}, i < 3$. We thus obtain,

$$\varphi_3(\theta^3) = 0, y_3(\theta^1) \in [\frac{v^M}{3}, \frac{v^H}{3}], \tag{10}$$

$$\varphi_i(\theta^1) = \frac{1}{2}, y_i(\theta^1) = -\frac{y_1(\theta^1)}{2}, i < 3. \tag{11}$$

Consider θ^2. In order for individual 2 not to misrepresent her valuation to v^H, i.e., to θ^1, it must be that,

$$\frac{1}{2}v^M + y_2(\theta^1) \leq y_2(\theta^2) = y_3(\theta^2), \tag{12}$$

where the last equality follows by symmetry. Therefore, by strict budget balance

$$y_1(\theta^2) = -2(\theta^2) \geq -\frac{2}{3}v^M, \tag{13}$$

where the last inequality follows from (10), (11), and (12), since the most negative transfer from 1 at θ^2 is attained when $y_3(\theta^1) = \frac{v^M}{3}$ in which case $y_1(\theta^2) = -\frac{2}{3}v^M$.

Consider θ^4. If at θ^4 individual 1 misreports her valuation to θ^2, then she obtains $v^M + y_1(\theta^2) \geq \frac{1}{3}v^M = u_1(\theta^4)$. Hence, it must be that $y_1(\theta^2) = \frac{2}{3}v^M$.

Step 3. The allocation at θ^6 follows by a similar argument to that in Step 2, substituting θ^4 for θ^0, θ^5 for θ^1, and θ^{10} for θ^4.

Step 4. Recall that $\alpha = \beta = 0$. At θ^3, the allocation follows by considering a deviation of individual 2 to θ^1. At θ^7 the allocation follows by considering a deviation of individual 2 to θ^3; alternatively, since $\alpha = \beta = 0$, $\theta_1 = v^H$, and $\theta_3 = 0$, it must be that individual 2 obtains the same transfer regardless of whether she reports $\theta_2 = v^L$ or $\theta_2 = v^M$.

Step 5. At θ^9, the allocation follows by considering a deviation of individual 2 to either θ^7 or θ^3. Now observe that $y_2 = y_3 = \frac{1}{3}v^M$, hence $y_1 = -\frac{2}{3}v^M$, which is independent of whether individual 1 reports v^H, v^M, or v^L as in any of these cases she obtains the object so that her transfer must be unaffected. Therefore, since $v^H < \frac{2}{3}v^M$, the utility of 1 at $\theta_1 = v^L$ is negative, that is, $u_1 = v^H - \frac{2}{3}v^M < 0$, which contradicts the individual rationality.

By steps 1-5 there does not exist an anonymous mechanism satisfying the desiderata. The proof follows from Proposition 4. □

	θ_1	θ_2	θ_3	φ_1	φ_2	φ_3	t_1	t_2	t_3
θ^0	v^H	v^H	v^H	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^1	v^H	v^H	$\{v^M, v^L, 0\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^M}{6}$	$-\frac{v^M}{6}$	$\frac{v^M}{3}$
θ^2	v^H	v^M	v^M	1	0	0	$-\frac{2v^M}{3}$	$\frac{v^M}{3}$	$\frac{v^M}{3}$
θ^3	v^H	v^M	$\{0, v^L\}$	$1-\alpha$	α	0	$v^M\left(\alpha - \frac{2}{3}\right)$	$\frac{v^M}{3} - \alpha v^M$	$\frac{v^M}{3}$
θ^4	v^M	v^M	v^M	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^5	v^M	v^M	$\{v^L, 0\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^L}{6}$	$-\frac{v^L}{6}$	$\frac{v^L}{3}$
θ^6	$\{v^M, v^H\}$	v^L	v^L	1	0	0	$-\frac{2v^L}{3}$	$\frac{v^L}{3}$	$\frac{v^L}{3}$
θ^7	$\{v^M, v^H\}$	v^L	0	$1-\beta$	β	0	$-t_2 - t_3$	$(\alpha - \beta)\, v^L + \frac{v^M}{3} - \alpha v^M$	$\frac{v^L}{3}$
θ^8	v^L	v^L	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^L}{6}$	$-\frac{v^L}{6}$	$\frac{v^L}{3}$
θ^9	$\{v^L, v^H, v^M\}$	0	0	1	0	0	$-\frac{2v^M}{3} + 2\alpha v^M$	$\frac{v^M}{3} - \alpha v^M$	$\frac{v^M}{3} - \alpha v^M$
θ^{10}	v^L	v^L	v^L	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^{11}	0	0	0	0	0	0	0	0	0

Table 1.

Along similar arguments but with substantially more algebra and notation it can be shown that in this environment there does not exist a classically efficient mechanism in the general case when $N \geq 2$. Note that finiteness of the type space is not the root cause of the problem in this example: The problem is that the individuals who do not obtain the object cannot be sufficiently compensated by the sole highest-valuation winner of the object since not enough value is generated from the allocation to simultaneously satisfy incentive compatibility, budget balance, and individual rationality when the difference between the lowest non-zero value and 0 is too small. This problem only gets worse when types can take a continuum of values.

In contrast to the *interim* setting of [18], Proposition 2 implies that there does not exist a Pareto efficient way to dissolve a partnership in a robust setting. The reason is that if the individuals initially hold property rights then it is more difficult to satisfy the individual rationality constraints. This is true under symmetric and asymmetric scenarios alike, that is, when the initial shares are equal or not. A separate special case is when the object is initially owned by one of the individuals, similar to a resale of an object that had previously been acquired by one of the individuals in an auction. [22] It could be surmised that there ought to exist a feasible classically efficient decision rule akin to a second-price auction and Proposition 2 implies that is not the case.

We now construct an example of an anonymous robust efficient mechanism in this environment.[23] There are many ways in which Pareto efficiency may be relaxed while satisfying feasibility. An obvious possibility is to assign the good in some instances with a positive probability to the individual with the second-highest valuation, which is still more efficient than assigning the good to some other individual with a lower valuation. Assigning the good in this way directly reduces the payments that must be made by the highest-valuation individual when she wins the object, and reduces the payments indirectly by differentiating between the incentive constraints of the second-highest and lower-valuation individuals.

Proposition 3. *Let $N = 3$. Suppose that $\Theta_i = \{0, v^L, v^M, v^H\}$, $\forall i$, and $v^L < \frac{2}{3}v^M$. Then there is an anonymous robust efficient mechanism, which assigns the object to the second highest-valuation individual with probability $\alpha = \frac{1}{2v^M}(\frac{2}{3}v^M - v^L)$ when there are no ties. That mechanism is described in Table 1, setting $\beta = \alpha$.*

Proof. Again consider Table 1. The proof follows along the lines of the proof of Proposition 2 with no differences in Steps 1-3, and the following distinctions:

In Step 4, at θ^3, when $\varphi_2 = \alpha > 0$, then that lowers the necessary transfer to individual 2 (still determined from the deviation of 2 to θ^1; this in turn determines the transfer to individual 2 at θ^7, as noted in Table 1.

In Step 5, at θ^9 in order for the individual 2 to not deviate either to θ^3 or θ^7, it must be that,

$$y_2(\theta^9) \geq \max\{(\alpha - \beta)v^L + (\frac{1}{3} - \alpha)v^M, (\frac{1}{3} - \alpha)v^M\}$$

[22] [53] provides an example of a classically efficient auction with resale under the *interim* constraints. In the present case, however, the environment is slightly different in that the original owner does not value the object at zero, as would an auctioneer in a standard auction environment. It is indeed an implication of the results in [18] that even under the *interim* constraints there does not exist a classically efficient decision rule in the present case.

[23] Note that while incentive compatible, a VCG mechanism will not simultaneously satisfy *ex-post* individual rationality and budget balance, as in the previous example, so that VCG mechanisms are out of question here too.

By setting $(\frac{1}{3}-\alpha)v^M = v^L$ and $\beta \geq \alpha$, that ensures that at θ^9, the individual rationality constraint of 1 holds with equality, while minimizing α subject to the constraint that 2 would not want to misrepresent from θ^9 to either θ^3 or θ^7; by setting $\beta = \alpha$ the loss in the aggregate surplus is minimized also at θ^7 so that the mechanism cannot be Pareto dominated by any mechanism which satisfies incentive compatibility along with all the other constraints.

Finally, to assure incentive compatibility, we must set $\varphi_i = 0$ at $\theta = (0,0,0)$, which has no effect on Pareto efficiency. □

5 Robustness, *ex ante*, *interim* constrained efficiency

In their formulation of robust mechanism design, [9] provide a framework for mechanism design in the context of rich type spaces.[24] In this section, we use that framework to define *ex ante* and *interim* notions of constrained efficiency for such robust considerations. We then show that in any commonly studied environment, these notions coincide with robust efficiency defined in Section 2.

As in [9], a type space is a collection,

$$\mathcal{T} = (T_i, \hat{\theta}_i, \hat{\pi}_i)_{i=1}^n, \text{ where,}$$

$t_i \in T_i$ is individual i's type, $\hat{\theta}_i : T_i \to \Theta_i$, so that $\hat{\theta}_i$ is i's payoff type when his type is t_i, and $\hat{\pi}_i : T_i \to \bar{T}_{-i}$, so that $\hat{\pi}_i(t_i)$ is the hierarchy of i's beliefs when his type is t_i. To sum up, the economy is now completely specified by the list,

$$\Gamma = (N, Y, \mathcal{T}, u),$$

and Γ is assumed to be common knowledge. In addition, in each state t, each individual i knows her private information t_i. Given a type space $\mathcal{T}$ we again define a decision rule d as a mapping $d : T \to \bar{Y}$. Note that the formulation of Section 2 can be embedded in this more general framework: let $\mathcal{T}$ be the payoff type space, i.e., $\mathcal{T}_i \equiv \Theta_i$. On the other hand,

[24] See, e.g., [35] and [11] for a general definition and discussion of rich type spaces.

a decision rule d on $\boldsymbol{\Theta}$ naturally induces a decision rule d_T on T.As before, the set of all decision rules is given by $\mathcal{D}$.

In this setting, we can imagine different *ex ante* and *interim* notions of (constrained) efficiency. One can then vary the type space and ask what decision rules are constrained efficient in a sense that is robust to such variation. To define the domination relations we fix the type space $\mathcal{T}$.[25]

Given a type space $\mathcal{T}$ and a decision rule d, the individual i's corresponding *ex ante* and *interim* expected utilities are given by,

$$U_i^{\mathcal{T}}(d) = \int_{t \in \mathcal{T}} u_i(d(t), \hat{\theta}(t)) \mathbf{d}\hat{\pi}_i(t),$$

$$U_i^{\mathcal{T}}(d \mid t) = \int_{t_{-i} \in T_{-i}} u_i(d(t), \hat{\theta}(t)) \mathbf{d}\hat{\pi}_i(t_{-i} \mid t_i).$$

Definition 5. *A decision rule d' ex ante dominates d on $\mathcal{T}$, denoted $d' \blacktriangleright^{\mathcal{T}} d$, iff,*

$$U_i^{\mathcal{T}}(d') \geq U_i^{\mathcal{T}}(d), \quad \forall i \in N, \tag{14}$$

with at least one strict inequality; d' interim dominates d on $\mathcal{T}$, denoted $d' \triangleright^{\mathcal{T}} d$, iff,

$$U_i^{\mathcal{T}}(d' \mid t) \geq U_i^{\mathcal{T}}(d \mid t), \quad \forall t_i \in T_i \forall i \in N, \tag{15}$$

with at least one strict inequality.

We remark that the notion of *ex post* domination on $\mathcal{T}$ is defined as in Section 2 and denoted by $\succ^{\mathcal{T}}$.

For our purposes, it will be enough to limit attention to two sorts of type spaces: *all full support common prior payoff type spaces*, defined as in [9]; and *all full support subjective*

[25]In the following definitions it is implicitly assumed that the type space $\mathcal{T}$ is finite, that is that each T_i is finite. This will be enough for our purposes here. The definitions can easily be extended to more general type spaces, e.g., the universal type space, as long as T is a Hausdorff space, which is the case as long as $\boldsymbol{\Theta}$ is compact, see [35].

priors payoff type spaces, which we introduce here.[26] A type space $\mathcal{T}$ is a payoff type space if $T_i = \Theta_i$ and $\hat{\theta}_i$ is the identity map, $\forall i$. A payoff type space satisfies a *full support common prior* assumption if in addition $\exists p \in int(\bar{\boldsymbol{\Theta}})$, such that,

$$\hat{\pi}_i(t_i)[t_{-i}] = p(t_{-i} \mid t_i), \;\; \forall t_i \in T_i, \forall i \in N.$$

A payoff type space satisfies a *full support subjective priors*[27], if, $\exists (p_1, \ldots, p_n) \in int(\bar{\boldsymbol{\Theta}})^n$, such that,

$$\hat{\pi}_i(t_i)[t_{-i}] = p_i(t_{-i} \mid t_i), \;\; \forall t_i \in T_i, \forall i \in N.$$

We maintain the assumption of full support throughout so that when we refer to, e.g., a subjective prior payoff type space, we really mean a full support subjective prior payoff type space and no confusion should arise. We also remark that the set of all subjective prior payoff type spaces is, of course, a superset of the set of all common prior payoff type spaces. From now on we will consider either of the two kinds of payoff type spaces.

A decision rule d on a payoff type space $\mathcal{T}$ is naturally a decision rule on $\boldsymbol{\Theta}$. Moreover, if a given (constrained) set of decision rules $D \subset \mathcal{D}$ is feasible for all subjective priors payoff type spaces, then it is sensible to think of D as a set of feasible decision rules on $\boldsymbol{\Theta}$. This will be the case if all the constraints specifying the decision rules in D are satisfied (or not) on all the subjective priors payoff type spaces, for example, all the *ex post* constraints specified in Section 2 evidently have this property. For such decision rules and feasible sets D all the definitions of Section 2 apply.[28] When considering all subjective priors payoff type spaces, heuristically, each individual does not know the others' subjective priors. We now fix a

[26] One could also consider the universal type space, where T_i is the set of all i's *coherent* hierarchies of beliefs, see also [35] or [11]. Indeed, our definition of robust efficiency is applicable to the *universal type space*, and our main result of this section, Theorem 2 below, holds on the universal type space *a fortiori*.

[27] To formally represent a subjective priors type space in the belief hierarchy, where each individual has a subjective prior over the payoff types, and these subjective priors are common knowledge, such a type space would be infinite and different from the payoff type space. Nevertheless, such a subjective priors type space is homeomorphic to the representation given here.

[28] For a general type space $\mathcal{T}$ it is not true that every decision rule d on T is also a decision rule on $\boldsymbol{\Theta}$ – two type profiles $t, t' \in T$ may pertain to the same profile of payoff types, and d may assign different allocations to t and t'. As for the feasibility of decision rules, one could also take various sorts of constraints that do depend on the specific type space, derive the corresponding type-space specific feasible sets of decision rules, and then consider the intersection of these sets over the relevant type spaces. See [9] and also [30] for an extended discussion of this issue, especially regarding the incentive compatibility constraints.

$D \subset \mathcal{D}$ and assume that it is feasible for all subjective priors payoff type spaces.

Definition 6. *A decision rule d' uniformly ex ante dominates d, if $d' \blacktriangleright^{\mathcal{T}} d$, on all common prior payoff type spaces $\mathcal{T}$; we denote $d' \blacktriangleright^{CP} d$. A $d \in D$ is uniform ex ante constrained efficient in D if $\nexists d' \in D$, s.t., $d' \blacktriangleright^{CP} d$.*

A decision rule d' uniformly interim dominates d, if $d' \triangleright^{\mathcal{T}} d$, on all common prior payoff type spaces $\mathcal{T}$; we denote $d' \triangleright^{CP} d$. A $d \in D$ is uniform interim constrained efficient in D if $\nexists d' \in D$, s.t., $d' \triangleright^{CP} d$.

We denote by $D^{\blacktriangleright}$ and $D^{\triangleright}$ the sets of uniform *ex ante* and *interim* constrained efficient decision rules in D, respectively.

Definition 7. *A decision rule d' robust ex ante dominates d, if $d' \blacktriangleright^{\mathcal{T}} d$, on all subjective priors payoff type spaces $\mathcal{T}$; we denote $d' \blacktriangleright^{SP} d$. A $d \in D$ is weak ex ante constrained efficient in D if $\nexists d' \in D$, s.t., $d' \blacktriangleright^{SP} d$.*

A decision rule d' robust interim dominates d, if $d' \triangleright^{\mathcal{T}} d$, on all subjective priors payoff type spaces $\mathcal{T}$; we denote $d' \triangleright^{SP} d$. A $d \in D$ is weak interim constrained efficient in D if $\nexists d' \in D$, s.t., $d' \triangleright^{SP} d$.

We denote by $\tilde{D}^{\blacktriangleright}$ and $\tilde{D}^{\triangleright}$ the sets of weak *ex ante* and *interim* constrained efficient decision rules in D, respectively.

The more difficult it is for a decision rule to dominate another decision rule, the larger the set of undominated decision rules. Therefore, the following relationships hold:

$$D^{\blacktriangleright} \subset \tilde{D}^{\blacktriangleright} \subset \tilde{D}^{\triangleright} \subset D^* \text{ and } D^{\blacktriangleright} \subset D^{\triangleright} \subset \tilde{D}^{\triangleright} \subset D^*. \tag{16}$$

Theorem 2. *A decision rule d' ex post Pareto dominates d, if and only if, d' uniformly ex ante dominates d.*

Proof. It is evident that if d' *ex post* Pareto dominates d, then d' uniformly *ex ante* dominates d. For the converse, suppose that d' does not *ex post* Pareto dominate d. For each $i \in N$, let $\mathbf{\Theta} = A_{i,d'} \cup A_{i,d} \cup A_{i,d'd}$, where $u_i(d'(\theta), \theta) > u_i(d'(\theta), \theta)$, $\forall \theta \in A_{i,d'}$, $u_i(d'(\theta), \theta) < u_i(d'(\theta), \theta)$, $\forall \theta \in A_{i,d}$, and $u_i(d'(\theta), \theta) = u_i(d'(\theta), \theta)$, $\forall \theta \in A_{i,d'd}$. Note that there exists at least one i

such that $A_{i,d} \neq \emptyset$. Now take $p \in int(\bar{\mathbf{\Theta}})$, such that p stacks most of the probability mass on $A_{i,d}$. Therefore,

$$\int_{\theta \in \mathbf{\Theta}} u_i(d'(\theta), \theta) \mathbf{d}p(\theta) < \int_{\theta \in \mathbf{\Theta}} u_i(d(\theta), \theta) \mathbf{d}p(\theta),$$

so that d' does not uniformly *ex ante* dominate d. □

Since the strongest and the weakest domination relations coincide, we have the following corollary.

Corollary 3. *For a $D \subset \mathcal{D}$, all the sets in (16) are identical,*

$$D^{\blacktriangleright} = \tilde{D}^{\blacktriangleright} = D^{\triangleright} = \tilde{D}^{\triangleright} = D^*.$$

As far as *interim* implementability of decision rules is concerned, [9] suggest that the requirement that a decision rule be implementable on all full support common prior payoff type spaces is a minimal requirement for robustness. If this criterion is applied to the question of efficiency, the minimal requirement for robustness is that a decision rule d be uniformly *interim* constrained efficient, that is, $d \in \tilde{D}^{\triangleright}$. By Theorem 2 and Corollary 3 this equivalent to requiring that d be robust efficient in D.

6 Concluding remarks

A quintessential question in the discussion of efficiency is whether and how the individuals may change from, or settle on, various decision rules. As described by [27] in the context of *interim* incentive efficient decision rules, the issue is two-fold: on the one hand, there may exist an informational state such that the allocation prescribed by a given decision rule (or, *a decision*) is Pareto dominated by some other allocation; on the other hand, unanimous agreement to change to a different decision rule would constitute a common knowledge that all individuals prefer the latter decision rule. [27] show that *interim* incentive efficiency is equivalent to non-existence of a common knowledge event, whereupon the individuals would unanimously prefer another feasible decision rule over a given feasible decision rule. Our definition of robust efficiency is not immediately amenable to such considerations – in the language of Section 2 it is not obvious how to define a common knowledge event. However, the definition of robust *interim* constrained efficient decision rules is naturally conducive to these issues.

Given a subjective priors payoff type space $\mathcal{T}$, as in [27] an event $R \subset T \equiv \boldsymbol{\Theta}$ is common knowledge on $\mathcal{T}$ iff $R = R_1 \times \ldots \times R_n$, $R_i \subset T_i$, and,

$$p_i(t'_{-i} \mid t_i) = 0, \quad \forall t \in R, \quad \forall t' \notin R, \ \forall i. \tag{17}$$

An event R is robust common knowledge iff it is common knowledge on every subjective priors payoff type space. Intuitively, if R is robust common knowledge, then, as long as the individuals' (rich) types are in R, all the individuals assign zero probability to the types outside R. regardless of what subjective prior over the payoff type space each individual holds, .

Definition 8. *A decision rule d' robust interim dominates d within R, denoted $d' \rhd_R d$, iff, $R \neq \emptyset$ and,*

$$U_i^{\mathcal{T}}(d' \mid t) \geq U_i^{\mathcal{T}}(d \mid t), \quad \forall t \in R, \ \forall i, \tag{18}$$

with at least one strict inequality, for all subjective priors payoff type spaces $\mathcal{T}$.

Suppose now that each individual may have a variety of different subjective priors, that is, consider all subjective priors payoff type spaces. If at some point prior to the realization of their private information – their types – the individuals agree on the decision rule that they should use, then they should settle on some *weakly ex ante* constrained efficient decision rule. By Theorem 2, such decision rule would be robust efficient. Equivalently, this would be the case if the decision rule were at such prior stage proposed by a benevolent social planner without knowledge of the individuals' priors; by Corollary 3, this would be the case even if the individuals had some common prior unknown to the social planner. However, under the former decentralized interpretation, it seems more sensible to assume that the individuals' priors were subjective.

It seems equally sensible to assume that the individuals should only consider rules that satisfy some notion of incentive compatibility. [9] showed that if a decision rule were *interim* incentive compatible on every common prior payoff type space, then such decision rule would be *ex post* incentive compatible as defined in Section 2. Therefore, if the decision rule were *interim* incentive compatible on every subjective priors payoff type space, then it would be *ex post* incentive compatible, *a fortiori*. In a decentralized discussion the individuals should then consider decision rules satisfying *ex post* incentive compatibility as well as any other constraints dictated by the economic environment at hand.

The question now is whether the individuals could agree to change from a given robust efficient incentive compatible decision rule after their private information has been realized. The following theorem is analogous to the characterization of *interim* incentive efficient decision rules in [39].

Theorem 4. *An incentive compatible decision rule d is robust efficient if and only if there does not exist a robust common knowledge event R and an incentive compatible decision rule d', such that $d' \rhd_R d$.*

Proof. For sufficiency, let $R = T$ and the result follows by Corollary 3. For necessity, suppose that $d' \rhd_R d$, where d' is incentive compatible and R is a robust common knowledge event. Define

$$d^*(t) = \begin{cases} d'(t), & \text{if } t \in R, \\ d(t), & \text{if } t \notin R. \end{cases}$$

Evidently $d^* \rhd^{SP} d$, and moreover, d^* is incentive compatible. To see this, take an i and first consider $t_i \in R_i$. He would not want to misreport to any type in R_i since d' is incentive compatible. Now fix a prior p_i. Since d' is *ex post* incentive compatible, it is *interim* incentive compatible under the prior p_i, and since $d' \rhd_R d$, d' *interim* dominates d under the prior p_i so that i's average payoff under p_i is at least as high under d' than under d. Since d is also *interim* incentive compatible under p_i, it follows that given a prior p_i, i would not have any incentives to misreport outside R_i. Now consider the case when $t_i \notin R_i$. Then, since R is robust common knowledge, it follows that the $t_{-i} \notin R_{-i}$, so that the decision rule coincide with d, which is *ex post* incentive compatible. Therefore, for every prior p_i, d^* is *interim* incentive compatible so that it is *ex post* incentive compatible. This implies that d is not robust efficient, a contradiction. □

Theorem 4 suggests that if the individuals are engaged in a robust efficient decision rule d then they could not unanimously agree to change to another incentive compatible decision rule. That is true at least to the extent that it could not be common knowledge on all subjective priors type space that the individuals unanimously prefer a different decision rule. One could still posit that if each individual learned some additional information beyond his type, then all individuals might be willing to change to some other decision rule. On the one hand, under *ex post* incentive compatibility the individuals would have no incentives to misreport even if they knew *all* the private information in the economy. Furthermore, robust efficient decision rules are precisely those that cannot be improved upon in every informational state *and still satisfy incentive compatibility.* Therefore, our heuristic intuition is that at least under some reasonable specifications of decentralized deliberations, the answer to the above question is no, and we leave this as a conjecture.

Another approach is to consider additional characteristics of the economic environment such as symmetry. A Rawlsian argument would suggest that in a symmetric environment *ex ante* equal individuals should be prone to elect at least a symmetric lottery over robust efficient decision rules; under conditions of risk aversion, or under a *max-min* criterion, the individuals should presumably prefer a deterministic choice of an *anonymous* decision rule over such a lottery.[29] While our main concern here has been with the environments where classically efficient rules do not exist, that is, where $D^* \cap \mathcal{D}^* = \emptyset$, in some environments even imposing efficiency might still lead to a substantial indeterminacy regarding transfers, or *prices.* In a symmetric environment one way to resolve this indeterminacy is to consider symmetric, or more general anonymous decision rules.

Suppose that for each i, $\Theta_i = \Theta$, so that $\boldsymbol{\Theta} = \Theta^n$. Denote by $\Pi(n)$ the set of all per-

[29] Anonymity and symmetry are well-known axioms in the literature especially concerning issues of fairness, see e.g., [36] and [47]. Symmetry requires that two individuals with the same preferences initial endowments receive the same allocation. Anonymity requires that when names of individuals are permuted, so are their consumption bundles. To keep the language consistent with the literature we refer to anonymous decision rules on the one hand; and to retain the intuitive interpretation of a symmetric set we refer to symmetric environments and symmetric sets of decision rules on the other hand. While related to the anonymity axiom, these latter symmetry axioms are different in that they concern both, the environment and the set of decision rules, rather than a specific decision rule. In particular, non-anonymous decision rules may in a symmetric environment belong to a symmetric set of decision rules.

mutations of n elements. A $\pi \in \Pi(n)$ may denote either a permutation of the individuals in N, or the corresponding permutation of the elements of vector $\theta \in \Theta^n$, or the corresponding permutation of the individuals' transfers $y_N \in \mathbb{R}^n$; furthermore, $\pi(\theta)$ denotes the permuted vector θ and $\pi(i)$ denotes the image of an element $i \in N$ under permutation π. For a decision rule d and a permutation π denote by d^π the decision rule $d \circ \pi$, that is, $d^\pi(\theta) = d(\pi(\theta))$, $\theta \in \boldsymbol{\Theta}$.

Definition 9. *An environment is symmetric iff* $\Theta_i = \Theta, \forall i$, *and for every* $y \in Y$, *and every permutation* $\pi \in \Pi(n)$, *there exists a* $y_0' \in Y_0$ *such that* $u_{\pi(i)}(y_0', \pi(y_N), \pi(\theta)) = u_i(y, \theta)$, $\forall i, \forall \theta \in \boldsymbol{\Theta}$.

Both our examples of sections 3 and 4 correspond to symmetric environments.

Definition 10. *Let the environment be symmetric. A decision rule* $d \in \mathcal{D}$ *is anonymous iff, for every permutation* π *of* N,

$$u_i(d(\theta), \theta) = u_{\pi(i)}(d(\pi(\theta)), \pi(\theta)), \forall i, \forall \theta.$$

Definition 11. *The set of decision rules* D *is symmetric iff the environment is symmetric and* $d \in D \Rightarrow d^\pi \in D$, $\forall d \in D$, *for all permutations* π.

To illustrate the aforementioned indeterminacy of prices consider a symmetric environment. There, existence of classically efficient decision rules implies that there also exist anonymous classically efficient decision rules. While the assumption of symmetry is somewhat restrictive, it covers several classical allocation problems. In particular, it applies to our examples in sections 3 and 4.

Proposition 4. *Suppose the environment is symmetric, quasi-linear, and* $D \subset \mathcal{D}$ *is convex. Then there exists a classically efficient decision rule in* D *if and only if there exists an anonymous classically efficient decision rule in* D.

Proof. For sufficiency, an anonymous classically efficient decision rule is classically efficient. For necessity, take a $d \in D^* \cap \mathcal{D}^*$ and suppose d is not symmetric. For each $\pi \in \Pi(n)$, d^π is also classically efficient and $d^\pi \in D$. For suppose to the contrary that there existed a d'

such that d' Pareto dominated d^π; in that case $d'^{\pi^{-1}} = d' \circ \pi^{-1}$ would Pareto dominate d, a contradiction. Therefore, $d^\pi \in D^* \cap \mathcal{D}^*, \forall \pi \in \Pi(n)$. Now let $d^*(\theta) = \frac{1}{|N|!} \sum_{\pi \in \Pi(n)} d^\pi(\theta)$. By convexity $d^* \in D$. It is also evident that d^* is symmetric. Finally, recall our assumption that the decision rules in $\mathcal{D}$ cannot entail infinite subsidies, that is, $\sum_{i \in N} d_i(\theta) \leq K$, $\forall \theta \in \mathbf{\Theta}$, for all $d \in D$ and some constant K. Along along with quasi-linearity, this implies that every classically efficient decision rule in $\mathcal{D}$ must yield the same total sum of the individuals' utilities and any feasible decision rule yielding such a total sum of utilities is classically efficient. Hence, $d^* \in D^* \cap \mathcal{D}^*$. □

In a more adventurous pursuit, one could ask under what conditions does utilitarianism lead to an anonymous decision rule and and what are the distributive consequences of possible extensions of these approaches to asymmetric environments? These questions, beyond the scope of the present study, are of interest and are relevant to current discussions in public economics, inequality, and distribution of wealth.

A different and related set of questions bring us to the discussion of classical efficiency yet another time. Suppose that in a given allocation problem there exists a classically efficient decision rule. First, if the individuals start by using some incentive compatible decision rule, will the individuals be prone to change to the classically efficient rule? Second, as the economy becomes large, do robust efficient decision rules in some sense converge to classical efficiency? In our view answering these and similar questions would be of some interests in the discussion of implementability and robustness of Walrasian outcomes and we hope that the concept of robust efficiency might prove a useful tool for such future investigations.[30]

7 Appendix

For a given set of decision rules $D \subset \mathcal{D}$ denote by $\mathcal{U}[D] = \{U^d(.) \mid d \in D\} \subset \mathbb{R}^{|\mathbf{\Theta}| \times n}$, i.e., $\mathcal{U}[D]$ is the set of utility allocations for all types of all agents, arising from the decision rules in D. Clearly, D satisfies convexity if and only if $\mathcal{U}[D]$ is convex. Similarly, D is closed

[30] In the context of exchange economies, [23] show existence of *interim* decision rules that converge to classical efficiency. [8] show that in single-peaked domains the Walrasian correspondence is not implementable in dominant strategies, or strategy-proof.

if and only if $\mathcal{U}[D]$ is closed in $R^{|\boldsymbol{\Theta}|\times n}$. Finally, note that since transfers are bounded by assumption, and all the other sets are finite, $\mathcal{U}[D]$ is bounded in $R^{|\boldsymbol{\Theta}|\times n}$, for any D.

We first show that for any D satisfying convexity, the set $\mathcal{U}[D^*]$ satisfies a weaker property. For $a, a' \in \mathcal{U}[D^*]$, define $S(a, a') = \{\mu \in (0,1) \mid \mu a + (1-\mu)a' \in \mathcal{U}[D^*]\}$.

Lemma 1. *For each $a, a' \in \mathcal{U}[D^*]$, either $S(a, a') = (0,1)$ or $S(a, a') = \emptyset$.*

Proof. Take $a, a' \in \mathcal{U}[D^*]$ and let $d, d' \in D^*$ be the corresponding scf's. Define $d_\alpha \equiv \alpha d + (1-\alpha)d'$, for $\alpha \in [0,1]$. Let $\bar{S} = (0,1) \setminus S(a, a')$, i.e., $\bar{S}$ is the set of α's such that $\alpha d + (1-\alpha)d' \notin D^*$.

Assume that $\bar{\alpha} \in \bar{S}$, for some $\bar{\alpha}$, hence $\bar{S} \neq \emptyset$. Let $d' \succ d_{\bar{\alpha}}$. Now take convex combinations of d' and d to dominate all d_α, s.t. $\alpha \leq \bar{\alpha}$, and take convex combinations of d' and d' to dominate all d_α, s.t. $\alpha \geq \bar{\alpha}$. Thus, $\bar{S} \neq \emptyset \Rightarrow \bar{S} = (0,1)$. □

Proof of Theorem 1. To see that $D^{\mathcal{W}} \subset D^*$ assume that $\exists d \in D^{\mathcal{W}}$ which solves (2) for some $\lambda(.)$ and $\exists d' \in D^{\mathcal{W}}$, s.t., $d' \succ d$. By the definition of $\succ$, inserting d' into the optimization program (2), we see that its value is higher than that obtained from d, for all $\lambda(.)$ and all $Pr \in int(\bar{\boldsymbol{\Theta}})$, a contradiction.

For the converse, we proceed in 5 steps.

Step 1. $\mathcal{U}[D^{\mathcal{W}}]$ has empty interior in $R^{|\boldsymbol{\Theta}|\times n}$.

Assume the opposite and take an open ball $o(a, \epsilon) = \{a' \mid ||a - a'||_2 < \epsilon\} \subset \mathcal{U}[D^{\mathcal{W}}]$, for some $\epsilon > 0$. Let d be the scf corresponding to the point a. Taking $a + \frac{\epsilon}{2}(1, 1..., 1)$ and letting d' be the corresponding scf we obtain $d' \succ d$, a contradiction.

Step 2. $\mathcal{U}[D^{\mathcal{W}}]$ is closed. This follows immediately from D closed.

Step 3. Either $\mathcal{U}[D^{\mathcal{W}}]$ has empty interior relative to $\mathcal{U}[D]$, or $\mathcal{U}[D^{\mathcal{W}}] = \mathcal{U}[D]$.

This follows from convexity of $\mathcal{U}[D]$ and Lemma 1.

Step 4. Take a direction $\alpha \in R^{|\boldsymbol{\Theta}|\times n}$, $||\alpha||_2 = 1$, and let the correspondence $a(\alpha)$ be defined as the solution to the linear program,

$$a(\alpha) = \arg \max_{a \in \mathcal{U}[D]} \alpha.a,$$

where $\alpha.a$ is the standard scalar product between the two vectors. Then,

$$\mathcal{U}[D^{\mathcal{W}}] = cl(\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha)).$$

By compactness of $\mathcal{U}[D^{\mathcal{W}}]$, $a(\alpha) \neq \emptyset, \forall \alpha$. By convexity of $\mathcal{U}[D]$, it is clear that if $int_{R^{|\mathbf{\Theta}| \times n}}(\mathcal{U}[D]) \neq \emptyset$, then $bo(\mathcal{U}[D]) = \cup_{\alpha \in R^{|\mathbf{\Theta}| \times n}} a(\alpha)$, where $bo(.)$ denotes the boundary of the set, i.e., the set of all its limit points which are not in its interior; and if $int_{R^{|\mathbf{\Theta}| \times n}}(\mathcal{U}[D]) = \emptyset$, then $\mathcal{U}[D]) = \cup_{\alpha \in R^{|\mathbf{\Theta}| \times n}} a(\alpha)$. If $int_{\mathcal{U}[D]}(\mathcal{U}[D^{\mathcal{W}}]) = \emptyset$, then since $\mathcal{U}[D^{\mathcal{W}}]$ is closed (Step 2), $\mathcal{U}[D^{\mathcal{W}}] = cl(\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha))$.[31]
On the other hand, if $\mathcal{U}[D^{\mathcal{W}}] = \mathcal{U}[D]$, then by Step 1 and convexity of $\mathcal{U}[D]$, $\mathcal{U}[D^{\mathcal{W}}]$ is a compact and convex linear subset of $R^{|\mathbf{\Theta}| \times n}$. Hence there exists a $\bar{\alpha} \in R_{++}^{|\mathbf{\Theta}| \times n}$, s.t. $\mathcal{U}[D^{\mathcal{W}}] = a(\bar{\alpha})$, (this $\bar{\alpha}$ must be strictly positive by the definition of $\succ$), so that $\mathcal{U}[D^{\mathcal{W}}] = a(\bar{\alpha}) \subset \cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha) \subset \cup_{\alpha \in R^{|\mathbf{\Theta}| \times n}} a(\alpha) = \mathcal{U}[D] = \mathcal{U}[D^{\mathcal{W}}]$, and the claim follows.
Step 5. Fix weights $\lambda(\theta) = (\lambda_1(\theta), ..., \lambda_n(\theta))$. Consider the linear program (**??**),

$$\arg\max_{d \in D} \sum_{\theta \in \mathbf{\Theta}} \sum_{i \in N} \lambda_i(\theta) u_i(\theta) =$$

$$\arg\max_{a \in \mathcal{U}[D]} \sum_{\theta \in \mathbf{\Theta}} \sum_{i \in N} \lambda_i(\theta) a_{\theta,i}.$$

Now observe that $\{(\lambda_i(\theta))_{\theta \in \mathbf{\Theta}, i \in N} \mid \lambda \in R_{++}^{|\mathbf{\Theta}| \times n}\} = \{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}\}$, which proves the theorem.

□

We remark that fixing $\bar{Pr} \in int(\bar{\mathbf{\Theta}})$, then

$$\{\left(\lambda_{\theta,i} \bar{Pr}(\theta)\right)_{\theta \in \mathbf{\Theta}, i \in N} \mid \lambda \in R_{++}^{|\mathbf{\Theta}| \times n}\} = int(\{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}\}).$$

[31] Observe that even in this case one may construct examples such that $\mathcal{U}[D^{\mathcal{W}}] = \cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha)$, i.e., where $\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha)$ is closed. That is not the case whenever $bo(\mathcal{U}[D])$ is a smooth manifold.

On the other hand,

$$\{(\lambda_i Pr(\theta))_{\theta\in\mathbf{\Theta},i\in N} \mid \lambda \in R^n_{++}, Pr \in int(\bar{\mathbf{\Theta}})\} \neq \{\alpha \in R^{|\mathbf{\Theta}|\times n}_{++}\}.$$

Observe also that if either the assumption of full support, $Pr \in int(\bar{\mathbf{\Theta}})$, or the assumption that all the welfare weights must be strictly positive is dropped, then one can construct examples where $D^{\mathcal{W}}$ is a strict subset of $\mathcal{D}^*$.

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