Robustness and Efficiency

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Abstract

The notion of *ex post* constrained efficiency is appropriate for efficiency considerations in environments where a desideratum is robustness to informational specification. It is a generalization of *ex post* incentive efficiency defined by Holmstrom and Myerson [4]. In an example of monopsony, it is shown that even when there exist *ex post* efficient allocation rules, there might still be *ex post* constrained efficient allocation rules that are not *ex post* efficient. *Ex post* constrained efficiency can also be defined via maximization of a social welfare criterion, or as a robust *ex ante* or *interim* welfare criterion over individuals' rich types.

1 Introduction

It is well known that in strategic environments with incomplete information, Pareto optimal decision rules might sometimes fail to exist. This is the case, for example, in a bargaining situation, where a buyer and a seller have private information regarding their reservation values, or when a group of individuals must decide whether or not to provide a public good without any external subsidies and when no individual is required to contribute beyond how much they privately value the good.[1] In a nutshell, the problem is the socially optimal allocation requires the individuals to reveal their private information, and in order to provide

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[1]See [9] for the bargaining situation, [6] for the public good provision, and for example [5] for a unified treatment.

the individuals with the incentives to do so, some of the social surplus must be foregone. Often, an additional desideratum on decision rules is that they are robust to the details of the specification of individuals' private information – either because it might be implausible to think that such details could at all be ascertained with the precision implied by the model, or the individuals might allow for a possibility that other individuals' beliefs over uncertain parameters belong to a richer set. Under such additional desideratum, it is even more difficult that Pareto efficiency might be obtained. Rather then insisting on Pareto efficiency, it is natural to ask what decision rules are in some sense the most efficient given the constraints of the economic environment.[2]

In this note we discuss efficiency of robust social choice functions, or social decision rules in environments with incomplete information. Similar to strategic considerations under robustness, efficiency considerations should also be independent from the details of the informational specification . A simple example is when a decision rule yields an efficient outcome for every realization of the individuals' private parameters, that is, when the decision rule is in fact *ex post* Pareto efficient. Then, the distribution over private parameters, or how the individuals thought the parameters were distributed, does not affect efficiency considerations. A natural extension to environments where *ex post* Pareto efficient rules do not exist is *ex post* constrained (Pareto) efficiency. A decision rule is *ex post* constrained efficient if, given the constraints of the economic environment, there does not exist an alternative decision rule, such that this alternative decision rule makes all individuals weakly better off for all possible vectors of private parameters (and at least one individual strictly better off for some private parameters). Again, if an alternative decision rule dominated a given decision rule in this way, it would dominate it in expected terms, no matter how the private parameters were distributed. The notion of *ex post* constrained efficiency thus results from a natural distribution-free extension of Pareto comparison of feasible decision rules.

Holmstrom and Myerson [4] defined a version of *ex post* constrained efficiency under the *interim* (Bayesian) incentive constraints, which they called the *ex post* incentive efficiency. They then discarded this notion on the grounds that at the *ex post* stage all the private information should presumably be revealed and the informational constraints should

[2] See for instance [10], who focus on efficiency in the case of providing a public good.

then not matter when making efficiency comparisons.[3] The definition of *ex post* constrained efficiency in this paper is Homlmstrom and Myerson's definition with general constraints on decision rules. This minor restatement of their definition then makes it quite transparent why the notion makes sense regardless of whether the constraints on the decision rule are of informational or some other nature. In order for an alternative decision rule to be viable in the first place, it must satisfy the constraints of the environment so that there can be an efficiency comparison with other rules that similarly satisfy the constraints. , for example, the rules satisfying the incentive constraints. The *ex post* constrained efficient decision rules are therefore those that are undominated within some feasible subset of all decision rules.[4]

It is quite difficult for a decision rule to dominate another decision rule in this *ex post* sense. For example, when the distribution over private parameters is known, in order for a decision rule to dominate another decision rule *ex ante*, it might still make some individuals worse off for some draws of private parameters. This suggests a different approach to defining efficiency criteria that are robust to the specification of the prior. In order to dominate a given decision rule, the alternative decision rule must do so on a rich set of the individuals' priors. That is, one can specify a rich type space of individuals' beliefs over the uncertain parameters, and then require that the decision rule dominates another decision rule for some set of such type spaces either at the *ex ante* or the *interim* stage.[5] The more general the type spaces that are allowed, the more difficult it will be to dominate a decision rule, and the weaker the resulting concept of constrained efficiency. The concept of *ex post* constrained efficiency is the weakest possible as whether a decision rule dominates another or not is entirely independent of the individuals' beliefs over the type space. We therefore define two notions of *ex ante* and *interim* constrained efficiency: when the set of type spaces is given by all common prior type spaces over the payoff types, and when it is given by the payoff type space where the individuals may have any beliefs over the payoff types. The strongest notion is given by *ex ante* domination on all common prior type spaces over the payoff types. In our Theorem 2 we show that this coincides with *ex post* domination described above, so

[3]That is, they argued that the *ex post* Pareto efficiency should then be the appropriate efficiency concept.

[4][2] propose a full ranking of mechanisms, based on a *min-max* criterion.

[5]See [7] and [3] for a definition of rich type spaces. We then follow [1], who apply this construction to mechanism design.

that any such robust efficiency concept is therefore equivalent to the *ex post* constrained efficiency.

Constraints that define the set of viable decision rules are typically given by incentive compatibility, individual rationality, budget balance, and so on. We apply *ex post* constrained efficiency to a standard example of a monopsonist auctioneer, who wants to acquire an item from a number of sellers with private costs of producing the item. It is well known that in such an environment there exist *ex post* efficient allocation rules, e.g., a second-price auction where the monopsonist acquires the item from the seller with the lowest quote at the price of the second-lowest quote [6] One could then surmise that any *ex post* constrained efficient allocation rule must be *ex post* efficient, i.e., that these allocation rules differ in transfers that different types of individuals obtain. However, that is not the case: we give an example of a *ex post* constrained efficient allocation rule in this environment, which is not efficient: the good is provided by a specific seller at the monopsonist's value of the item (which is known), independently of the realization of the private parameters. The interpretation is that when a specific individual, in this case one of the sellers, hijacks the allocation rule, there is no reason to expect this allocation rule to be efficient. Moreover, there is no alternative feasible allocation rule, which would compensate that individual in every state of the world.

The paper is structured as follows. In Section 2 we give the definition of *ex post* constrained efficiency, give the example of the monopsony, and show that *ex post* constrained efficient decision rules can also be obtained as maximizers of a social welfare function, Theorem 1.[7] In Section 3, we give *ex ante* and *interim* versions of robust efficiency criteria and then show that these coincide with *ex post* constrained efficiency. In Section 4 we give the proof of Theorem 1.

[6]Given the considerations of robustness, we also consider the incentive and other feasibility constraints at the *ex post* stage, i.e., in the case of private values this is equivalent to dominant-strategy incentive compatibility. Second-price auction goes back to [11], and following [8] there has been a large literature focusing on auctions that are optimal for a monopolist seller, which in the case here would be an auctions that is optimal for the monopsonist buyer.

[7]Theorem 1 follows [4] and is also related to work by [12].

2 Ex post constrained efficient decision rules

There are n individuals, $i \in N = \{1, ..., n\}$. For each individual i there is a finite set of *payoff types*, Θ_i, and $\Theta = \Theta_1 \times ... \times \Theta_n$. We denote by $\bar{\Theta}$ the set of probability measures over Θ. There is a set of (deterministic) feasible allocations Y, and $\bar{Y}$ is the set of randomized allocations, i.e., $\bar{Y}$ is the set of all probability measures over Y. We will assume that Y is given by a finite number of different alternatives in Y_0 and the vectors of transfers between the individuals $y \in Y_1 \times ... \times Y_n$, so that $Y_i = \mathbb{R}$, where $y_i > 0$ when i receives a payment, and $y_i < 0$ when i makes a payment.[8]

Given a $\theta \in \Theta$, individual i's preferences are given by a von Neumann-Morgenstern utility function $u_i(., \theta) : \bar{Y} \to \mathbb{R}$, where $u_i(\bar{y}, \theta)$ is linear in $\bar{y}$, since $\bar{y}$ is a probability measure. The payoff parameter θ_i incorporates all the i's payoff relevant private information – θ_i is not known by other individuals – and the functions $u_i(., .)$ are common knowledge. In general, we assume that u_i is increasing in the transfer y_i, for each $y \in Y$, and for each $\theta \in \Theta$. One may also make a more restrictive assumption of quasi-linearity.[9] For the purpose of this section, it is common knowledge that the individuals' types lie in Θ, but the individuals do not know the probability distribution over these parameters.[10] In an environment with private values, where i's preferences are affected only by his own private parameter θ_i, this has a standard interpretation where $u_i(., \theta), \theta \in \Theta$ can then be imagined as the domain of i's preferences. A decision rule, or a social choice function, is a mapping $\delta : \Theta \to \bar{Y}$. We denote by $\mathcal{D}$ the set of all decision rules.

The following notion of *ex post* Pareto domination and the corresponding frontier of constrained efficient decision rules is our main focus. This notion embodies the idea that the distribution over the individuals' types is not known: in order for a decision rule to improve over another decision rule, it must be an improvement for every realization of parameters,

[8]For example, N may be a set of sellers and buyers of some number of objects, in which case Y is the set of possible allocations of the objects and the vector of transfers between the individuals. Different allocations may entail different production costs, and so on.

[9]That is, that the utility functions are additively separable in the allocation and the monetary transfer, quasi-linear with respect to transfers, and are normalized so that 1 unit of money is worth 1 unit of utility to all agents, i.e., $u_i(y, \theta) = y_i + \nu_i(y_0, \theta)$.

[10]See Section 3 for a description in the context of rich type spaces, which uses the framework provided by [1].

and it must be a strict improvement for some realizations of parameters. In general, the decision rules under consideration might be restricted by some feasibility constraints, such as incentive compatibility, which determine a subset $D \subset \mathcal{D}$. In that case, in order to dominate a given decision rule, the dominating decision rule must also satisfy these same constraints.

Definition 1. *A decision rule γ ex post dominates δ, denoted by $\gamma \succ \delta$, if,*

$$u_i(\gamma(\theta), \theta) \geq u_i(\delta(\theta), \theta), \ \forall \theta \in \Theta, \forall i \in N, \tag{1}$$

with a strict inequality for at least one θ and at least one i. Given a $D \subset \mathcal{D}$, a decision rule $\delta \in D$ is ex post constrained efficient in D, or undominated *in D, if there does not exist a $\gamma \in D$, such that, $\gamma \succ \delta$. We denote by $D^e \subset D$, the set of all ex post constrained efficient rules in D.*

This *ex post* dominance relationship has been considered by [4] when the set of decision rules is restricted by *interim* incentive constraints, a concept which they called *ex post incentive efficiency.* They discarded *ex post* incentive efficiency as unavailing on the grounds that the individuals would at the *ex post* stage know all the information, so that *ex post* efficiency should presumably be the appropriate concept. However, when the set D is interpreted as describing some feasibility constraints that the decision rules must satisfy, then the *ex post* dominance relationship describes which of these rules would be unanimously discarded by the individuals when the distribution over types is not known, i.e., the point-wise Pareto-dominance relationship. It would hardly make sense for the individuals to discard a feasible decision rule on the grounds that it is dominated by a decision rule, which is not feasible.[11]

[11] For example, by the revelation principle the *interim* incentive compatible decision rules are those that would be attainable in a Bayes Nash equilibrium of *some* extensive game form describing the allocation problem at hand. The set of *ex post* incentive efficient allocation rules, as defined in [4], are then the *interim* incentive compatible decision rules, which are not dominated by any other *interim* incentive compatible decision rule. That, of course, is quite different from the *interim* incentive compatible decision rules, which are not dominated by any arbitrary decision rule, i.e., the *interim* incentive compatible decision rules, which are *ex post* efficient. First, the set of *ex post* efficient and *interim* incentive compatible decision rules might be empty. Second, an incentive compatible decision rule should not be discarded because it is dominated by a rule, which is not incentive compatible – the alternative decision rule would never be attainable in a Bayes Nash equilibrium of any extensive form game and would therefore not be feasible in the first place.

When the set of decision rules is in no way constrained, that is, when $D = \mathcal{D}$, then the set of such undominated decision rules $\mathcal{D}^e$ is the set of all Pareto efficient decision rules (or *ex post* efficient decision rules; [9] also call such rules *classically efficient*). Of course such unconstrained efficient decision rules might fail to satisfy equilibrium, participation, and other feasibility constraints. Examples of such constraints are incentive compatibility, individual rationality, budget balance, and so on, which we state next. These constraints then determine the admissible set of decision rules $D \subset \mathcal{D}$.

Decision rule $\delta \in D \subset \mathcal{D}$ is *ex post* incentive compatible, if,

$$u_i(\delta(\theta), \theta) \geq u_i(\delta(\theta_{-i}, \theta_i'), \theta), \forall \theta_i' \in \Theta_i, \forall \theta \in \Theta, \forall i \in N. \tag{2}$$

Given a $D \subset \mathcal{D}$, we denote by D^* the set of *ex post* incentive compatible decision rules in D; $\mathcal{D}^*$ is the set of all *ex post* incentive compatible decision rules.

We denote by D^{*e} the set of *ex post* constrained efficient incentive compatible decision rules, which contains all the *ex post* constrained efficient rules in D^*. In environments with transfers, the set D may be constrained also by budget balance, individual rationality, and market clearing. Such restrictions seem appropriate especially in market settings, or any other settings where the individuals presumably cannot be forced to engage in transactions they dislike, and there is no external subsidizing entity. In the remainder of this section, we assume that $Y = Y_0 \times_{i \in N} Y_i$, i.e., the environment is one with transfers.

Decision rule $\delta \in D \subset \mathcal{D}$ satisfies *ex post individual rationality* if there exists a $y_0 \in Y_0$, such that, $u_i(\delta(\theta); \theta) \geq u_i(y_0, 0, ..., 0; \theta)$, $\forall \theta \in \Theta$. Usually, $u_i(y_0, 0, ..., 0; \theta)$ is normalized to 0. A decision rule therefore satisfies *ex post* individual rationality if there exists an alternative that allows the individuals to opt out and attain some security level of utility. For example, in a market setting, y_0 is the outcome where no transactions of any commodities take place, i.e., each individual can opt out and not trade (which may also affect the other individuals' utilities).[12]

[12] An environment with transferrable utilities is separable without the individual rationality constraints, but is no longer separable when the individual rationality constraints apply, see also [1].

Decision rule δ satisfies *ex post budget balance*, if,[13]

$$\sum_{i \in N} \delta(\theta)[\omega]_i \leq 0, \forall \omega \in \Omega, \forall \theta \in \Theta. \tag{3}$$

A decision rule therefore satisfies *ex post* budget balance if, in any state, no external subsidies are required to carry out all the transfers between the individuals.

Y_0 is a production-and-consumption set if $Y_0 \subset \mathbb{Z}^{N \times L}$, where L is the number of commodities, and $Y_{0,i,\ell}$ is the set of production or consumption possibilities of commodity $\ell \in L$ by individual i. Decision rule δ satisfies *ex post market clearing*, if, Y_0 is a production-and-consumption set, the budget constraint (3) holds with equality, and,[14]

$$\sum_{i \in N} \delta(\theta)[\omega]_{0,i,\ell} = 0, \forall \ell \in L, \forall \omega \in \Omega, \forall \theta \in \Theta. \tag{4}$$

A decision rule satisfies market clearing if the markets for all commodities and transfers balance exactly. Note that all of the above constraints are given by linear inequalities, so that the set D resulting from these constraints is convex.

Example 1. *Consider an environment with transfers where $Y_0 = N$, Θ_i is given by positive integers $\{1, 2, ..., K\}$, where $K > 1$, $\Theta_n = \{K\}$, and $Y_i = \mathbb{R}$, $i \in N$. The utility of $i \in \{1, ..., n-1\}$ is given by $u_i(y, \theta) = y_i - 1_{\{y_0 = i\}}\theta_i$, $y = (y_0, y_1, ..., y_n)$. This is a setting of monopsony, where agents $1, ..., n-1$ are sellers, who can produce one unit of a good at a privately known cost θ_i, and agent n is the buyer, who wants to acquire one unit of the good that he values at K. A decision rule determines what seller produces the good, if $y_0 \in \{1, 2, ...n-1\}$ (or by convention, $y_0 = n$, if no transaction takes place), as well as the transfers of all agents in the economy. Assume that the set of decision rules is $D^* \subset \mathcal{D}^*$, given by the individually rational, budget balanced and ex post incentive compatible decision rules. In this example we assume that $N = \{1, 2, 3\}$, so that there are 2 sellers and individual*

[13] Recall that the decision rule prescribes a lottery $\bar{y} = \delta(\theta)$ over $\bar{Y} = \Delta(\times_{i \in \{0\} \cup N} Y_i)$, for each θ, so that $\delta(\theta)[\omega]_i \in Y_i$ is the transfer to i when payoff types are θ and the realization of the lottery is determined by $\omega \in \Omega$.

[14] More generally, one must be able to represent Y_0 as a production-and-consumption set. One could also think of the case where there is no excess demand, i.e., free disposal is allowed, in which case the inequality in (4) is weak.

3 is the buyer.

First consider the decision rule δ^* whereby the sellers submit their quotes and the buyer acquires the good from the seller who submitted the lowest quote at the price of the second lowest quote. That is, the buyer effectively runs a second-price auction, so that $\delta^* \in D^* \cap \mathcal{D}^e$. Therefore $\delta^* \in D^{*e}$, that is, δ^* is an ex post constrained efficient decision rule given D^* since it is ex post efficient, individually rational, budget balanced, and incentive compatible.

Now consider the constant decision rule δ^1 whereby the first seller produces the good and hands it to the buyer at the price K no matter what the sellers' production costs are, i.e., $d^1(\theta) = (1, K, 0, -K), \forall \theta \in \Theta$. Evidently, $\delta^1 \in D^*$, and it is also quite evident that $\delta^1 \notin \mathcal{D}^e$: it is ex post dominated, for example, by the following decision rule:

$$\delta'(\theta) = \begin{cases} (2, K-\theta_1, \theta_1, -K), & \text{if } \theta_2 < \theta_1 \\ (1, K, 0, -K), & \text{if } \theta_1 \geq \theta_2 \end{cases}$$

Thus, δ' is a decision rule whereby the lowest-cost seller, i, provides the good to the buyer; if the lowest cost seller is 2, then he receives the payment equal to the quote of the second lowest-cost seller, θ_1, the buyer pays K, and seller 1 receives the payment of $K - \theta_1$; if the lowest cost seller is 1, then he receives the payment of K and produces the good that he hands over to the buyer. However, $\delta' \notin D^*$ because d' is not incentive compatible: for example, when $\theta_2 = 1$ and $\theta_1 > \theta_2$, seller 1 would have incentives to understate his cost in order to lower the amount of the payment received by seller 2 and thus increase the payment to himself. Therefore, δ' does not ex post dominate δ^1 given D^*.

Indeed, $\delta^1 \in D^{*e}$. To see that, suppose there was a $\gamma \in D^*$, such that $\gamma \succ \delta^1$. Observe first that whenever $\theta_1 < \theta_2$, γ must equal δ – otherwise not enough surplus would be generated to compensate individual 1. Similarly, when $\theta_1 = \theta_2$, γ and δ must allocate the same utilities to all agents, so that in terms of incentive and feasibility properties γ and δ are indistinguishable on the set $\theta_1 \leq \theta_2$.

Now take a $\theta_2 < \theta_1$. Suppose that γ assigns the production of the good to agent 2. The payment to 1 must be at least $K - \theta_1$ and the payment to 2 must be at least θ_2 – since γ dominates δ and by individual rationality of 2. In particular, when $\theta_2 = \theta_1 - 1$,

$y_1(\theta_1, \theta_2) \in [K - \theta_1, K - \theta_1 + 1]$, $y_2(\theta_1, \theta_2) \in [\theta_1 - 1, \theta_1]$, and $y_3(\theta_1, \theta_2) \in [K - 1, K]$. Now take $\theta_2' < \theta_2 - 1$ and by incentive compatibility, the payment to 2 must remain as before, $y_2(\theta_1, \theta_2') = y_2(\theta_1, \theta_2)$, which implies that all the other payments must remain unchanged. Let $\theta_1' = \theta_2' + 1$, and $y_1(\theta_1', \theta_2') \in [K - \theta_1', K - \theta_1' + 1]$, so that $y_2(\theta_1', \theta_2') \in [\theta_1' - 1, \theta_1']$ and by incentive compatibility of 1, $y_1(\theta_1, \theta_2') = y_1(\theta_1', \theta_2')$, so that $y_2(\theta_1, \theta_2') \in [K - \theta_1', K - \theta_1' + 1]$. But then when 1's types is θ_1', when 2 is of type θ_2, 2 would have incentives to misreport to θ_2'.

The decision rule δ^1 represents a situation where seller 1 "hijacks" the decision rule and allocates the maximal possible payment from the buyer to himself without assuming any further knowledge regarding the distribution over types. δ^1 maximizes the revenue of seller 1 point-wise in the type space, under these constraints. Any other decision rule, which yields at least as high a utility to all types of all agents, violates one of the constraints. In the above example there exist *ex post* efficient decision rules in D^*, e.g., δ^*, but not all decision rules in D^{*e} are *ex post* efficient, e.g., d^1. Thus, there *may still be constrained efficient rules in* D^* that are not Pareto efficient – no decision rule in D^{*e} can simultaneously improve the well being of all types of all individuals. Note that all the decision rules in the above example satisfy market clearing.

A different approach to defining optimal incentive compatible decision rules is by way of maximization of a social welfare function. Given a $D \subset \mathcal{D}$, a decision rule $\delta \in D^*$ is a robust welfare maximizer if,

$$\begin{gathered} \exists \lambda : \Theta \to R^N_{++}, \text{ s.t.,} \\ \delta \in \arg\max_{\gamma \in D^*} \textstyle\sum_{\theta \in \Theta} \sum_{i \in N} \lambda_i(\theta) u_i(\delta(\theta), \theta). \end{gathered} \tag{5}$$

Denote by $D^{*r} \subset D^*$ the set of robust welfare maximizers. In the following theorem, denote the closure of a set O by $cl(O)$.

Theorem 1. *Let $D \subset \mathcal{D}$ be closed and convex. Then $cl(D^{*r}) = D^{*e}$.*

As mentioned above, individual rationality, budget balance, or market clearing assure that D is a closed and convex set.

3 Ex ante and interim optimality under robustness

In their formulation of robust mechanism design, [1] provide a framework for mechanism design in the context of rich type spaces.[15] In this section, we use that framework to define *ex ante* and *interim* notions of constrained efficiency for such robust considerations. We then show that in any commonly studied setting, these notions coincide with the *ex post* constrained efficiency defined in Section 2.

As in [1], a type space is a collection,

$$\mathcal{T} = (T_i, \hat{\theta}_i, \hat{\pi}_i)_{i=1}^n, \text{ where,}$$

$t_i \in T_i$ is individual i's type, $\hat{\theta}_i : T_i \to \Theta_i$, so that $\hat{\theta}_i$ is i's payoff type when his type is t_i, and $\hat{\pi}_i : T_i \to \bar{T}_{-i}$, so that $\hat{\pi}_i(t_i)$ is the hierarchy of i's beliefs when his type is t_i. For our purposes, it will be enough to limit our attention to two sorts of type spaces: the *payoff type space*, $\mathcal{T}^{\Theta}$, where $T_i = \Theta_i$, $\forall i \in N$; and the *common prior payoff type spaces* $\mathcal{T}^{\Theta,CP}$, where additionally, $\exists Pr \in int(\bar{\Theta})$, such that,[16]

$$\hat{\pi}_i(t_i)[t_{-i}] = Pr(t_{-i} \mid t_i), \ \forall t_i \in T_i, \forall i \in N.$$

To sum up, the economy is now completely specified by a list,

$$\Gamma = (N, Y, \mathcal{T}, u),$$

and all these components of Γ are assumed to be common knowledge. In addition, in each state t, the individual i knows his private information t_i. Given a type space $\mathcal{T}$ we now define a decision rule δ as a mapping $\delta : \Theta \to \bar{Y}$. Note that the formulation of Section 2 can be embedded in this more general framework: set $\mathcal{T}$ the payoff type space, i.e., $\mathcal{T}_i \equiv \Theta_i$. As before, the set of all decision rules is given by $\mathcal{D}$.

[15]See, e.g., [7] and [3] for a general definition and discussion of rich type spaces.

[16]Recall that we assume the common prior has a full support. One could also consider the universal type space, where T_i is the set of all i's *coherent* hierarchies of beliefs, see e.g., [7] or [3]. Indeed, our definition of robust optimality is applicable to the *universal type space*, and our result holds on the universal type space *a fortiori*.

Within this framework, one can imagine several possible *ex ante* and *interim* notions of optimality in a robust setting. We define four such notions, two *ex ante* and two *interim* notions, depending on the type spaces considered.

Given a type space $\mathcal{T}$, and a $\delta \in D \subset \mathcal{D}$, a decision rule $\gamma \in D$ *ex ante* dominates δ on $\mathcal{T}$, if,

$$\int_{\theta\in\Theta} u_i(\gamma(t), \hat{\theta}(t))d\hat{\pi}_i(t) \geq \int_{\theta\in\Theta} u_i(\delta(t), \hat{\theta}(t))d\hat{\pi}_i(t), \ \forall i \in N, \tag{6}$$

with a strict inequality for at least one i. We denote $\gamma \succ_a^{\mathcal{T}} \delta$.

Definition 2. *Given a $D \subset \mathcal{D}$, we say that γ ex ante dominates δ on all common prior payoff type spaces if $\gamma \succ_a^{\mathcal{T}} \delta$, for all common prior common prior payoff type spaces $\mathcal{T}$; We say that a decision rule $\delta \in D$ is uniform ex ante constrained efficient, if there does not exist a $\gamma \in D$, such that γ ex ante dominates δ on all common prior payoff type spaces. We denote by $\tilde{D}_a$ the set of uniform ex ante constrained efficient decision rules.*
We say that a decision rule $\delta \in D$ is weak ex ante constrained efficient, if there does not exist a $\gamma \in D$, such that $\gamma \succ_a^{\mathcal{T}^\Theta} \delta$, that is, if δ is not ex ante dominated in D on the payoff type space $\mathcal{T}^\Theta$. We denote by D_a the set of weak ex ante constrained efficient decision rules.

The analogous *interim* notions of optimality are defined as follows. Given a type space $\mathcal{T}$, and a $\delta \in D \subset \mathcal{D}$, a decision rule $\gamma \in D$ *interim* dominates δ on $\mathcal{T}$, if,

$$\int_{t_{-i}\in T_{-i}} u_i(\gamma(t), \hat{\theta}(t))d\hat{\pi}_i(t_{-i} \mid t_i) \geq \int_{t_{-i}\in T_{-i}} u_i(\delta(t), \hat{\theta}(t))d\hat{\pi}_i(t_{-i} \mid t_i), \ \forall \theta_i \in \Theta_i \forall i \in N, \tag{7}$$

and there exists at least one $i \in N$ and $t_i \in T_i$, such that the inequality is strict. We denote $\gamma \succ_{in}^{\mathcal{T}} \delta$.

Definition 3. *Given a $D \subset \mathcal{D}$, we say that a decision rule γ interim dominates δ on all common prior payoff spaces, if $\gamma \succ_{in}^{\mathcal{T}} \delta$ for all common prior payoff type spaces $\mathcal{T}$.We say that a decision rule $\delta \in D$ is uniform interim constrained efficient, if there does not exist a $\gamma \in D$, such that γ interim dominates δ on all common prior payoff spaces. We denote by $\tilde{D}_{in}$ the set of uniform interim constrained efficient decision rules.*
We say that a decision rule $\delta \in D$ is weak interim constrained efficient, if there does not exist a $\gamma \in D$, such that $\gamma \succ_{in}^{\mathcal{T}^\Theta} \delta$, that is, if δ is not interim dominated in D on the payoff

type space $\mathcal{T}^{\Theta}$. We denote by D_{in} the set of uniform interim constrained efficient decision rules.

These notions can be related to the assumptions regarding how the individuals settle upon the decision rule itself. In the *ex ante* case, before their engagement in the allocation problem, the individuals can settle on whether a specific decision rule $\delta \in D$ should be used or whether there might be a viable alternative decision rule. The set D is the set of decision rules, which are in some sense feasible (e.g., incentive compatible, individually rational, and so forth). The first possibility is that the individuals are certain that once they engage in the allocation problem, they will have some common prior over the payoff type space but they do not know what that common prior will be – i.e., they will only learn that common prior once they have engaged in the allocation problem. If there exists a decision rule $\gamma \in D$, which *ex ante* dominates δ on all common prior payoff type spaces, then it could be viably suggested to the individuals (by one of the individuals, or by some external agency) that rather than δ, γ should be used. None of the individuals should then have any reason to raise an objection, and some individuals should strictly prefer γ at least in some states. Given absent exogenous or payoff irrelevant considerations, it seems hardly plausible that the individuals should then use δ. The second possibility is that the individuals are certain that once they engage in the allocation problem they will only care about payoff relevant information (but might for whatever reason not necessarily have a common prior over the payoff relevant information). Then, if there exists a decision rule $\gamma \in D$, which *ex ante* dominates δ on the payoff type spaces, the individuals should again embrace such a suggestion. Therefore, if the individuals have an option to change the decision rule before their engagement in the allocation problem, depending on whether or not the individuals will have a common prior over the payoff types or not, it is the *weak* or *uniform ex ante* constrained efficient decision rules that are viable.

Similarly, the *interim* constrained efficient decision rules are those where the individuals would not have any reason to change the decision rule at the *interim* stage, when each individual's private information has been realized. When such a change is proposed by some external agency, e.g., the mechanism designer who does not know what beliefs the individuals might hold but knows that the individuals hold beliefs only over payoff relevant information, then the two *interim* notions obtain.

Note that the more difficult it is for a decision rule to dominate another decision rule, the larger the set of undominated decision rules. Therefore, for a given set of decision rules D, the following relationships hold:

$$\tilde{D}_a \subset D_a \subset D_{in} \subset D^e \text{ and } \tilde{D}_a \subset \tilde{D}_{in} \subset D_{in} \subset D^e. \tag{8}$$

Theorem 2. *A decision rule γ ex post dominates δ, if and only if, γ ex ante dominates δ on all common prior payoff type spaces.*

Proof. It is evident that if γ *ex post* dominates δ, then γ weakly *ex ante* dominates δ. For the converse, suppose that γ does not *ex post* dominate δ. For each $i \in N$, let $\Theta = A_{i,\gamma} \cup A_{i,\delta} \cup A_{i,\gamma\delta}$, where $u_i(\gamma(\theta), \theta) > u_i(\gamma(\theta), \theta)$, $\forall \theta \in A_{i,\gamma}$, $u_i(\gamma(\theta), \theta) < u_i(\gamma(\theta), \theta)$, $\forall \theta \in A_{i,\delta}$, and $u_i(\gamma(\theta), \theta) = u_i(\gamma(\theta), \theta)$, $\forall \theta \in A_{i,\gamma\delta}$. Note that there exists at least one i such that $A_{i,\delta} \neq \emptyset$. Now take Pr_δ, s.t., Pr_δ stacks most of the probability mass on $A_{i,\delta}$. Therefore,

$$\int_{\theta \in \Theta} u_i(\gamma(\theta), \theta) dPr_\delta(\theta) < \int_{\theta \in \Theta} u_i(\delta(\theta), \theta) dPr_\delta(\theta),$$

so that γ does not *ex ante* dominate δ on all common prior payoff type spaces. □

Since the strongest and the weakest domination relations coincide, we have the following corollary.

Corollary 3. *For a $D \subset \mathcal{D}$, all the sets in (8) are identical, that is,*

$$\tilde{D}_a = D_a = \tilde{D}_{in} = D_{in} = D^e.$$

One might be tempted to define an even weaker notion of domination where a decision rule δ is not dominated if it is not dominated on any common prior payoff type space. That is, define δ to be *strong* robust *ex ante* constrained efficient, if, given a $D \subset \mathcal{D}$, and for any common prior payoff type space $\mathcal{T}$, there does not exist $\gamma \in D$, such that, γ dominates δ on $\mathcal{T}$. However, it seems that such strong robust *ex ante* constrained efficiency might be too restrictive leading to issues with existence – often, there will exist *ex post* constrained efficient decision rules δ and γ such that the individuals will be better off in δ for some draw of types, and in γ for some other draw of types. In that case, there will exist a common prior type space $\mathcal{T}$ such that δ will dominate γ on $\mathcal{T}$, and $\mathcal{T}'$ such that γ will dominate δ on $\mathcal{T}'$ an neither δ nor γ will be strong *ex ante* constrained efficient. Therefore, strong *ex ante* constrained efficiency might lead to issues with the existence of such decision rules.

4 Proof of Theorem 1

For a given set of decision rules $D \subset \mathcal{D}$ denote by $\mathcal{U}[D] = \{U^\delta(.) \mid \delta \in D\} \subset \mathbb{R}^{|\Theta| \times n}$, i.e., $\mathcal{U}[D]$ is the set of utility allocations for all types of all agents, arising from the decision rules in D. Clearly, D satisfies convexity if and only if $\mathcal{U}[D]$ is convex. Similarly, D is closed if and only if $\mathcal{U}[D]$ is closed in $R^{|\Theta| \times n}$. Finally, note that since transfers are bounded by assumption, and all the other sets are finite, $\mathcal{U}[D]$ is bounded in $R^{|\Theta| \times n}$, for any D.

We first show that for any D satisfying convexity, the set $\mathcal{U}[D^*]$ satisfies a weaker property. For $a, a' \in \mathcal{U}[D^*]$, define $S(a, a') = \{\mu \in (0,1) \mid \mu a + (1-\mu)a' \in \mathcal{U}[D^*]\}$.

Lemma 1. *For each $a, a' \in \mathcal{U}[D^*]$, either $S(a, a') = (0,1)$ or $S(a, a') = \emptyset$.*

Proof. Take $a, a' \in \mathcal{U}[D^*]$ and let $\delta, \delta' \in D^*$ be the corresponding scf's. Define $\delta_\alpha \equiv \alpha\delta + (1-\alpha)\delta'$, for $\alpha \in [0,1]$. Let $\bar{S} = (0,1) \backslash S(a, a')$, i.e., $\bar{S}$ is the set of α's such that $\alpha\delta + (1-\alpha)\delta' \notin D^*$.

Assume that $\bar{\alpha} \in \bar{S}$, for some $\bar{\alpha}$, hence $\bar{S} \neq \emptyset$. Let $\tilde{\delta} \succ \delta_{\bar{\alpha}}$. Now take convex combinations of $\tilde{\delta}$ and δ to dominate all δ_α, s.t. $\alpha \leq \bar{\alpha}$, and take convex combinations of $\tilde{\delta}$ and δ' to dominate all δ_α, s.t. $\alpha \geq \bar{\alpha}$. Thus, $\bar{S} \neq \emptyset \Rightarrow \bar{S} = (0,1)$. □

Proof of Theorem 1. To see that $D^{r*} \subset D^{**}$ assume that $\exists \delta \in D^*$ which solves (5) for some $\lambda(.)$ and $\exists \tilde{\delta} \in D^*$, s.t., $\tilde{\delta} \succ \delta$. By the definition of $\succ$, inserting $\tilde{\delta}$ into the optimization program (5), we see that its value is higher than that obtained from δ, for all $\lambda(.)$ and all $Pr \in int(\bar{\Theta})$, a contradiction.

For the converse, we proceed in 4 steps.

Step 1. $\mathcal{U}[D^*]$ has empty interior in $R^{|\Theta| \times n}$.

Assume the opposite and take an open ball $o(a, \epsilon) = \{a' \mid ||a - a'||_2 < \epsilon\} \subset \mathcal{U}[D^*]$, for some

$\epsilon > 0$. Let δ be the scf corresponding to the point a. Taking $a + \frac{\epsilon}{2}(1, 1..., 1)$ and letting $\tilde{\delta}$ be the corresponding scf we obtain $\tilde{\delta} \succ \delta$, a contradiction.

Step 2. $\mathcal{U}[D^*]$ is closed. This follows immediately from D being closed.

Step 3. Either $\mathcal{U}[D^*]$ has empty interior relative to $\mathcal{U}[D]$, or $\mathcal{U}[D^*] = \mathcal{U}[D]$.

This follows directly from convexity of $\mathcal{U}[D]$ and Lemma 1.

Step 4. Take a direction $\alpha \in R^{|\Theta| \times n}$, $||\alpha||_2 = 1$, and let the correspondence $a(\alpha)$ be defined as the solution to the linear program,

$$a(\alpha) = \arg \max_{a \in \mathcal{U}[D]} \alpha.a,$$

where $\alpha.a$ is the standard scalar product between the two vectors. Then,

$$\mathcal{U}[D^*] = cl(\cup_{\alpha \in R_{++}^{|\Theta| \times n}} a(\alpha)).$$

By compactness of $\mathcal{U}[D^*]$, $a(\alpha) \neq \emptyset, \forall \alpha$. By convexity of $\mathcal{U}[D]$, it is clear that if $int_{R^{|\Theta| \times n}}(\mathcal{U}[D]) \neq \emptyset$, then $bo(\mathcal{U}[D]) = \cup_{\alpha \in R^{|\Theta| \times n}} a(\alpha)$, where $bo(.)$ denotes the boundary of the set, i.e., the set of all its limit points which are not in its interior; and if $int_{R^{|\Theta| \times n}}(\mathcal{U}[D]) = \emptyset$, then $\mathcal{U}[D]) = \cup_{\alpha \in R^{|\Theta| \times n}} a(\alpha)$. If $int_{\mathcal{U}[D]}(\mathcal{U}[D^*]) = \emptyset$, then since $\mathcal{U}[D^*]$ is closed (Step 2), $\mathcal{U}[D^*] = cl(\cup_{\alpha \in R_{++}^{|\Theta| \times n}} a(\alpha))$.[17]

On the other hand, if $\mathcal{U}[D^*] = \mathcal{U}[D]$, then by Step 1 and convexity of $\mathcal{U}[D]$, $\mathcal{U}[D^*]$ is a compact and convex linear subset of $R^{|\Theta| \times n}$. Hence there exists a $\bar{\alpha} \in R_{++}^{|\Theta| \times n}$, s.t. $\mathcal{U}[D^*] = a(\bar{\alpha})$, (this $\bar{\alpha}$ must be strictly positive by the definition of $\succ$), so that $\mathcal{U}[D^*] = a(\bar{\alpha}) \subset \cup_{\alpha \in R_{++}^{|\Theta| \times n}} a(\alpha) \subset \cup_{\alpha \in R^{|\Theta| \times n}} a(\alpha) = \mathcal{U}[D] = \mathcal{U}[D^*]$, and the claim follows.

Step 5. Fix weights $\lambda(\theta) = (\lambda_1(\theta), ..., \lambda_n(\theta))$. Consider the linear program (**??**),

$$\arg \max_{\delta \in D} \sum_{\theta \in \Theta} \sum_{i \in N} \lambda_i(\theta) u_i(\theta) =$$

[17] Observe that even in this case one may construct examples such that $\mathcal{U}[D^*] = \cup_{\alpha \in R_{++}^{|\Theta| \times n}} a(\alpha)$, i.e., where $\cup_{\alpha \in R_{++}^{|\Theta| \times n}} a(\alpha)$ is closed. That is not the case whenever $bo(\mathcal{U}[D])$ is a smooth manifold.

$$\arg\max_{a\in\mathcal{U}[D]} \sum_{\theta\in\Theta}\sum_{i\in N} \lambda_i(\theta) a_{\theta,i}.$$

Now observe that $\{(\lambda_i(\theta))_{\theta\in\Theta,i\in N} \mid \lambda \in R_{++}^{|\Theta|\times n}\} = \{\alpha \in R_{++}^{|\Theta|\times n}\}$, which proves the theorem.
□

I remark that fixing $\bar{Pr} \in int(\bar{\Theta})$, then

$$\{\left(\lambda_{\theta,i}\bar{Pr}(\theta)\right)_{\theta\in\Theta,i\in N} \mid \lambda \in R_{++}^{|\Theta|\times n}\} = int(\{\alpha \in R_{++}^{|\Theta|\times n}\}).$$

On the other hand,

$$\{(\lambda_i Pr(\theta))_{\theta\in\Theta,i\in N} \mid \lambda \in R_{++}^{n}, Pr \in int(\bar{\Theta})\} \neq \{\alpha \in R_{++}^{|\Theta|\times n}\}.$$

Observe also that if either the assumption of full support, $Pr \in int(\bar{\Theta})$, or the assumption that all the welfare weights must be strictly positive is dropped, then one can construct examples where $D^*]$ is a strict subset of $\mathcal{D}^{r*}$. □

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