Optimal Robust Bargaining Games

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Abstract

We consider a bilateral-trade problem with incomplete information and risk-averse traders; utility functions are common knowledge but reservation values are private information. We define a *Mediated Bargaining Game* - a continuous-time double auction with a hidden order book. It is the *optimal* bargaining game in the sense that its *ex-post* Nash equilibria constitute the Pareto-optimal frontier of the set of all *ex-post* Nash equilibria of all bargaining games. In the Mediated Bargaining Game, Bayesian equilibria coincide with *ex-post* Nash equilibria. The inefficiency due to incomplete information is manifested through delay. As risk aversion of at least one agent tends to infinity, some equilibria tend to full efficiency. Under assumptions on agents' utilities, there is a unique equilibrium in which prices are linear in agents' types. Such prices are quite different from those imputed by the Nash bargaining solution. This equilbrium has a simple closed-form expression. Our approach is suitable for applications, such as wage bargaining between a firm and a worker.

1 Introduction

Bargaining between two impatient traders is a fundamental problem of economics. Since Rubinstein's [1982] result on the alternating-offers game with perfect information, many economists have been concerned with providing a similarly effective

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and tractable framework for settings with imperfect or incomplete information where agents' reservation values are private information. By the Myerson and Satterthwaite [1983] impossibility theorem, the outcomes are then necessarily inefficient, which is different from the perfect-information setup.

Most of the incomplete-information bargaining literature has focused on characterizing Bayesian equilibria of different versions of the alternating-offers game, with risk-neutral agents. This approach has three shortcomings. Such games have many Bayesian equilibria, which are hard to characterize. [1] In Bayesian approaches, agents are assumed to have precise knowledge of the distribution of each others' reservation values, or types. The effect of risk aversion on equilibrium contracts may matter, and in many relevant situations agents have different levels of risk aversion. Examples of such situations are wage bargaining between a worker and a firm, a large block-trade between a market maker and an investor, a real-estate trade, and bilateral peace talks. In the absence of adequate mechanism-design machinery, such contracts have been studied in the literature, but generally without imposing incentive constraints.

In this paper, we study robust equilibria for bargaining games in environments where agents are impatient and can be risk averse. We assume that the players' utility functions are common knowledge but that their reservation values are private. By robustness we mean that both the equilibrium concept and its efficiency are robust to traders' beliefs. We thus require an equilibrium of such a game to be an *ex-post* equilibrium, and that its outcome be no worse, in the Paretian sense, than any *ex-post* equilibrium outcome of any bargaining game. We call this equilibrium efficiency requirement *ex-post* constrained efficiency.

We define the Mediated Bargaining Game as a continuous-time double-auction in which the Mediator prevents traders from seeing each other's bids until the time of agreement.[2] The agreed price is then made public, and the trade takes place. The key feature of the Mediated Bargaining Game is that the information flow between the agents is minimized, so that agents recognize the surplus only upon agreement, when the game is over. The main result of our analysis is that the Mediated Bargaining Game is the optimal robust bargaining game.

We characterize mediated equilibria of the Mediated Bargaining Game. A mediated equilibrium is an *ex-post* Nash equilibrium in undominated and type-monotone strategies. We show that the set of outcomes of Mediated Equilibria under risk neutrality is dense in the set of outcomes of *ex-post* constrained-efficient equilibria of *all* dynamic bargaining games. Due to incomplete information, delay arises endogenously as a part of every equilibrium, and delay causes the inefficiency. All equilibrium outcomes of the Mediated Game are *ex-post* individually rational to both

[1] See Cramton [1984], Cho [1990], and Ausubel and Denekere [1992].

[2] A version of the Mediated Bargaining Game where the set of possible prices is finite was first proposed by Jarque, Ponsatí and Sákovics [2003]. In our model the prices are not restricted.

traders. Under risk neutrality all Mediated Equilibria have simple closed-form expressions, but this is true only for special cases of risk aversion. Nonetheless, when agents are risk averse or discount the future at different rates, mediated equilibria are constrained efficient. For a rich set of environments with risk aversion, there exists a unique linear equilibrium with a simple closed form, suitable for embedding in other models.

Under risk aversion there is risk-sharing in equilibrium, and if an agent becomes more risk averse, the outcomes become closer to full efficiency. The effect on equilibrium outcomes is the same if agents are more impatient or if they are more risk averse, i.e. an external observer cannot distinguish between a more impatient and a more risk-averse agent just by observing an equilibrium outcome. Under perfect information, this is also a feature of the subgame-perfect Nash equilibrium in the Rubinstein alternating-offers game.

The Mediated Bargaining Game is a decentralized game form implementing *ex-post* individually rational, incentive compatible, and constrained-efficient direct revelation mechanisms. Ledyard [1978] proves that if a game has an *ex-post* Nash equilibrium then the corresponding direct-revelation mechanism is *ex-post* incentive compatible.[3] *Ex-post* constrained efficiency of *ex-post* equilibria is equivalent to the *ex-post* constrained efficiency of an *ex-post* individually rational and incentive compatible direct revelation mechanism. Hence, for each mediated equilibrium, there is a direct revelation mechanism that is *ex-post* constrained efficient under *ex-post* incentive compatibility and individual rationality. Under risk neutrality, such mechanisms can be represented as probability distributions over posted prices (Theorem 1). Since the Mediated Bargaining Game implements *all* constrained-efficient mechanisms and nothing else, it is *the* optimal robust bargaining game.

As a model, direct revelation mechanisms are not equivalent to the indirect Mediated Bargaining Game. First, to construct a specific *ex-post* individually-rational and incentive-compatible mechanism, the designer has to know the agents' utility functions, but neither rationality nor preferences have to be common knowledge among the agents. In the Mediated Bargaining Game, the designer does not need to know anything about the agents, and just lets them play the game, but the preferences, rationality, and the equilibrium have to be common knowledge between the two traders. The second important difference is that the Mediated Bargaining Game is free of a commitment problems that affect the direct revelation mechanism. Generically, direct revelation mechanisms prescribe lotteries at the *ex-post* stage, and the agents can invert each lottery to figure out the type of the opposing player that they are facing. The assumption in a direct revelation mechanism is that agents can

[3] A mediated equilibrium *is not* a dominant-strategy equilibrium of the Mediated Bargaining Game, so that the relation to direct revelation mechanisms is not simply a consequence of the revelation principle. In fact, the Mediated Bargaining Game has no equilibria in dominant strategies.

commit to *ex-post* lotteries even though they can obtain something better once they know each other's types. But *ex-post* individual rationality requires that they cannot commit to *ex-post* payments, which will make them better off at the *interim* stage. Thus, *ex-post* individual rationality is inconsistent with commitment to *ex-post* lotteries. Such criticism does not apply to mediated equilibria because they specify deterministic trade every time it occurs, and time cannot be reversed to ameliorate the inefficiency resulting from delays.

As a final remark, we show that the set of mediated equilibria coincides with the set of separating perfect Bayesian equilibria of the Mediated Bargaining Game. Perfect Bayesian equilibria are outcome equivalent to Bayesian equilibria because out-of-equilibrium deviations cannot affect the updating of beliefs. Thus, imposing a weaker equilibrium notion and additional sequential rationality on the players does not change the set of equilibrium outcomes of the Mediated Bargaining Game, so that this set is robust to weaker equilibrium notions.

The Mediated Bargaining Game is the optimal robust bargaining game, and is observed in practice. One can interpret the Mediator as an order book that is closed. Several electronic exchanges, e.g. Nasdaq, Frankfurt, Stockholm, and others, allow for *hidden orders* which are put in the book but are not observable by other traders. The justification is that hidden orders are supposed to enhance efficiency, and to our knowledge there is no theoretical foundation in the existing literature. Albeit a very stylized model of exchange, our analysis of the Mediated Bargaining Game provides a strong theoretical support for that claim. Mediation is also widely used in conflict resolution, and practitioners point to the fact that effective mediation requires restricting direct information flows between the two parties.[4] That is precisely the defining feature of the Mediated Bargaining Game.

In Section 2 we review the literature. In Section 3 we give definitions of robust bargaining games and *ex-post* constrained efficiency. In Section 4 we review direct mechanisms under risk neutrality. In Section 5 we analyze the Mediated Bargaining Game under risk neutrality. In Section 6 we extend the results to risk aversion, and prove the existence of the unique linear mediated equilibrium. In Section 7, we show that Bayesian equilibria of the Mediated Bargaining Game are *ex-post* equilibria.

2 Related Literature

Our work is related to the literature on robust mechanism design, see Hurwicz [1972], Ledyard [1978], D'Aspremont and Gerard-Varet [1979], Neeman [2001], Chung and

[4]For example, Francesc Vendrell - the UN envoy to Central America, Namibia, Timor and Afganistan - says: "I prefer to negotiate separately with each party, rather than with both parties talking face to face." (El País (30/12/01). See also Dunlop [1984].

Ely [2003], Bergemann and Morris [2004] and Jehiel et al. [2005]. The recent literature studies *ex-post* implementation in common value problems, where *ex-post* implementation does not imply dominant-strategy direct mechanisms, which is different from our setting. Čopič and Ponsatí [2005] characterize *ex-post* individually-rational, incentive-compatible and constrained-efficient mechanisms for bilateral trade with risk aversion and generalize the results of Hagerty and Rogerson [1987]. The latter establish payoff equivalence to distributions over posted prices for a subclass of *ex-post* individually-rational and incentive-compatible mechanisms under risk neutrality.

Our work is also related to the literature on non-cooperative bargaining under incomplete information (see Ausubel, Cramton and Denekere [2002] for a survey). The closest are Cramton [1992] and Wang [2000]. Cramton [1992] extends the continuous-time game of Admati and Perry [1987] to two-sided uncertainty and constructs a separating equilibrium where trade occurs, with delay, whenever gains from trade exist. In the game of Wang [2000] there exists a class of outcome-equivalent separating *ex-post* equilibria. The outcome coincides with that of the linear mediated equilibrium under risk neutrality. In Example 8 we compare the efficiency (in *ex-ante* terms) of mediated equilibria with the equilibrium in Wang [2005] and Cramton [1992]. We show that mediated equilibria can dominate both of these, although the latter is not robust.

Comparison to cooperative bargaining is also relevant. The allocation in the linear mediated equilibrium coincides with the Nash bargaining solution (Nash [1950]) only when agents have the same risk aversion. Still, in asymmetric environments the risk sharing in the Nash bargaining solution goes in the same direction as in our model. But the approach of cooperative bargaining theory is nevertheless quite different. There is no incomplete information, incentives are not modeled explicitly, and outcomes are assumed to be efficient, so that there is no obvious way to model delay.

3 Dynamic bargaining games and robustness

The problem, preferences, and information structure. Two agents, a seller and a buyer $i = s, b$, bargain over the price $p \in [0, 1]$ of an indivisible good. The seller's cost of producing the good v_s, and the buyer's valuation of the good v_b are private information. We denote $v = (v_s, v_b)$. We assume that it is common knowledge that v is distributed according to a pdf G with a continuos density g, and support $supp(g) = [0, 1]^2$. We stress that common knowledge of the specific G is not necessary.

We assume that the agents are risk neutral, and they discount the future exponentially. In particular, when an agreement to trade at price p is reached on date $t \geq 0$, the seller's payoff upon trading at price p at t is $u_s(v_s, p, t) = e^{-t}(p - v_s)$ and the buyer's is $u_b(v_b, p, t) = e^{-t}(v_b - p)$. In Section 6 we relax these assumptions.

Robust equilibrium and efficiency requirements. Given some dynamic bargaining game form Γ, we impose that the equilibrium and efficiency notions be robust. Therefore, in equilibrium strategies and outcomes must be independent of beliefs, implying that the equilibrium be an *ex-post* equilibrium. We define the *ex-post* equilibrium and the robustness notion for a general dynamic bargaining game Γ, so that our definitions are abstract. In this Section we say nothing about the existence of such games. In Sections 5 we construct a dynamic bargaining game with robust equilibria that are optimal.

A dynamic bargaining game Γ is in our setup defined by the sets of traders' strategies, contingent on their type, the set of histories for each player, given past play of the game, and the outcome function, mapping strategy profiles into outcomes (i.e. terminal histories). Let $\mathcal{H}_t$ be the set of possible histories at time t. A strategy of player i is a mapping from his type v_i, time t, and history $h(t) \in \mathcal{H}_t$ into price bids,

$$p_i : [0, 1] \times [0, \infty) \times \mathcal{H}_t \to [0, 1] \ , \ i = s, b.$$

Each outcome is specified by a time $\tau(p_s, p_b, v_s, v_b)$ and a price at which trade occurs at that time, $\bar{p}(p_s, p_b, v_s, v_b)$. Note that if the trade never happens that is equivalent to $\tau(p_s, p_b, v_s, v_b) = \infty$.[5]

We say that strategies (p_s^*, p_b^*) constitute an *Ex-Post Nash equilibrium* (PEQ) if they are mutual best responses for each pair of types (v_s, v_b). More precisely, given the equilibrium strategy of say the buyer, p_b^*, the seller's strategy p_s^* satisfies:

$$p_s^* = \arg\max_{p_s} e^{-\tau(p_s, p_b^*, v_s, v_b)} \left(\bar{p}(p_s, p_b^*, v_s, v_b) - v_s \right), \forall v_s, v_b.$$

Denote $p^* = (p_s^*, p_b^*)$, and by $U_i(v; p^*)$ the equilibrium payoff to agent i, given strategies p^* and types v.

In a dynamic bargaining game Γ agents cannot be forced to trade at an unacceptable price, so that every equilibrium outcome must be individually rational to both agents.

The robust efficiency notion we impose is *ex-post* constrained efficiency [6] which

[5]For an extensive discussion of when games in continuous time are well-defined see Simon and Stinchcombe [1989]. For a discussion on admissible strategies and sensible outcomes in bargaining games with continuous-time see Sákovics[1993].

[6]This notion is related to *ex-post* incentive efficiency of direct revelation mechanisms, due to Holmstrom and Myerson [1983]. The difference is that *ex-post* incentive efficiency means *ex-post*

in the present context says the following. Take a dynamic bargaining game Γ and an PEQ $(p^*;\Gamma)$. We say that this PEQ is *ex-post* constrained efficient (PCE) if there does not exist another pair $(\tilde{p}^*;\Gamma')$, Γ' a dynamic bargaining game and $\tilde{p}^*$ an PEQ of Γ', such that

$$U_i(v;p^*) \leq U_i(v;\tilde{p}^*), \forall v \in [0,1]^2, i = s,b, \text{ and}$$

$$U_i(v;p^*) < U_i(v;\tilde{p}^*), \forall v \in V^{open} \subset [0,1]^2, \text{ for at least one } i.$$

We remark that a natural notion of equilibrium for dynamic games is the Perfect Bayesian Equilibrium (PBE), which employs a notion of sequential rationality. In general PBE need not be robust. In Section 6 we show that PBE of the dynamic game which we analyze in this paper are in fact robust.

By the revelation principle and Ledyard [1978], for each dynamic bargaining game Γ, and each *ex-post* equilibrium (p^*,Γ), there exists a direct revelation mechanism (mechanism) m, which is *ex-post* incentive compatible, and such that $U_i(v;p^*) = U_i^m(v), \forall v \in [0,1], i = s,b$, where $U_i^m(v)$ is the payoff to agent i in mechanism m, under truthful reporting.

4 Direct revelation mechanisms

In this Section we briefly review the results on *ex-post* incentive compatible (PIC), *ex-post* individually rational (PIR), and *ex-post* constrained-efficient (PCE) mechanisms (under PIR and PIC). The reader should note that PCE is imposed at the *ex-post* stage, and should not confuse this with the *ex-ante* or *interim* notions of optimality. The reader should also note that *ex-post* incentive compatibility implies equilibrium in dominant strategies and should not confuse that with the *interim* incentive compatibility which implies a Bayesian equilibrium. We also remark that PCE is weaker than either the *ex-ante* or *interim* constrained-efficiency notions (PCE is necessary for either of these two). PCE is the only notion among the three which is robust. For a more detailed discussion of the issues reviewed here see our companion paper Čopič and Ponsati [2005].

We first note the well known and simple fact that under risk-neutrality each PIRIC mechanism can be represented by a pair of functions $(\pi,\delta) : [0,1]^2 \to [0,1]^2$, where $\pi(v)$ is the price and $\delta(v)$ is the probability with which this price will obtain; with complementary probability no trade occurs, and the agents obtain 0 util-

constrained optimality of a mechanism given Bayesian incentive compatibility. Thus, it does not employ individual rationality, and incentive compatibility is imposed at the *interim* while PEQ is equivalent to PIC of the direct revelation mechanism. We use a different name in order to keep the distinction clear.

ity. We remark that in an PEQ (p^*, Γ), $\delta(v)$ corresponds to the shrinking of the surplus due to discounting, so that $\delta(v) = e^{-\tau(p^*(v))}$. Denote by $U_s^{\pi,\delta}(v_s', v_b; v_s) = \delta(v_s', v_b)(\pi(v_s', v_b) - v_s)$ the payoff to the seller under a mechanism (π, δ) when the reported types are (v_s', v_b) and seller's true type is v_s. Similarly for the buyer, $U_b^{\pi,\delta}(v_s', v_b; v_b) = \delta(v_s', v_b)(v_b - \pi(v_s', v_b))$. Also denote $U_i^{\pi,\delta}(v) = U_i^{\pi,\delta}(v_i, v_j; v_i), i, j \in \{s, b\}, i \neq j$. PIC and PIR of a mechanism $m = (\pi, \delta)$ are now formulated as follows:

$$U_s^{\pi,\delta}(v_i', v_j; v_i) \leq U_i^{\pi,\delta}(v), \forall v_i, v_i', v_j, i, j \in \{s, b\}, i \neq j; \; (P\text{IC})$$

$$\delta(v) > 0 \Rightarrow v_s \leq \pi(v) \leq v_b, \forall v. \; (P\text{IR})$$

The *ex-post* constrained efficiency under individual rationality (PCE) of mechanisms is formulated similarly as the PCE of a dynamic game. An PIRIC mechanism (π, δ) satisfies PCE if

$$\nexists(\pi', \delta'), \; P\text{IRIC and s.t. } U_i^{\pi,\delta}(v) \leq U_i^{\pi',\delta'}(v), \forall v, i = s, b, \text{ and}$$

$$U_i^{\pi,\delta}(v) < U_i^{\pi',\delta'}(v), \forall v \in V_0^{open} \subset [0,1]^2, \text{ for at least one } i.$$

The following Theorem also appears in the same form in Čopič and Ponsatí [2005].

Theorem 1. A mechanism (π, δ) is PCE if and only if there exists a probability distribution F_p, $supp(F_p) \subset [0, 1]$, such that $\delta(v) = F_p(v_b) - F_p(v_s)$ and $\pi(v) = E_{F_p}[p \mid v_s \leq p \leq v_b]$. Here $E_{F_p}[. \mid .]$ denotes the conditional expectation w.r.t. F_p.

Proof. See Appendix A. □

5 The Mediated Bargaining Game

In this Section we introduce the Mediated Bargaining Game (MBG), we define regular PEQ of the MBG, and we show that all of these are PCE. Thus, the MBG is an optimal robust bargaining game.

The game. The MBG is a dynamic double auction in continuous-time, with a Mediator. The Mediator is a dummy player whose only role is to receive bids, keep them secret while they are incompatible, and to announce the agreement as soon as it is reached. When the Mediator announces that an agreement has been reached, trade takes place at the agreed price, and the game ends.

In the MBG, the Mediator imposes a restriction on the agents' updating of beliefs. In particular, the agents can only update through the passing of time, and in an equilibrium, observable history at a time t is completely specified by t. The strategies therefore map types and times into bids, so that we can exclude h_t from the arguments of p_s and p_b.

There are two more rules of the MBG. First, the Mediator enforces commitment, so that the traders' strategies have to be weakly monotone w.r.t. time t (i.e. the seller can only decrease her bid at any moment, and the buyer can only increase it). This is also enough to have well-defined outcomes in MBG. Time-monotonicity can alternatively be thought of as a behavioral assumption. The second rule of the MBG requires the agents to move in a differentiable way w.r.t. time.

R1 $p_s(v_s, t)$ is (weakly) decreasing and $p_b(v_b, t)$ is (weakly) increasing, with respect to time.

R2 $p_i(v_i, t)$ are differentiable with respect to time, for all t.[7]

R1 and R2 imply that $\frac{\partial p_s(v_s,t)}{\partial t} \leq 0$ and $\frac{\partial p_b(v_b,t)}{\partial t} \geq 0$, $\forall v_i \in [0, 1]$.

We will restrict attention to undominated equilibria of MBG. In particular, for each PEQ profile p, a profile p' constructed by adding a *standstill interval* $[0, T)$, i.e. $p'_i(v_i, t + T) = p_i(v_i, t)$, is an PEQ as well, for any $T < \infty$. That is, as the opponent does not concede any positive amount until T, no concession prior to T is useful. Regardless of T, such strategy profiles p' are weakly dominated. We say that an PEQ is *undominated* if it does not have a standstill interval. We can similarly define undominated profiles under Bayesian Equilibrium (BE) and Perfect Bayesian Equilibrium (PBE) concepts (see Section 6 for precise definitions).

Mediated Equilibrium A *Mediated Equilibrium* (ME) is an PEQ of the MBG which is strictly type-monotone, undominated, and such that if $v_2 > v_1$, there $\exists t < \infty$ such that $p^*_s(v_s, t) = p^*_s(v_s, t)$.

In Appendix B we show that every Bayesian Equilibrium (BE) of the MBG has to be weakly type monotone (see Proposition 14). Since every PEQ is clearly a BE, this implies that all the PEQ have to be weakly type monotone. In the rest of this Section we characterize the ME of the MBG and show that they exist. We also show that the set of outcomes of ME is dense in the set of outcomes of PEQ of the MBG, so that the restriction to ME is purely for analytic convenience.

Proposition 2. A strategy profile p is a ME if and only if

1. $p_i(v_i, t)$, $i = s, b$ satisfy the first order conditions

$$\begin{aligned} (p_s(v_s, t) - v_s) &= \frac{\partial p_b(v_b,t)}{\partial t}, \\ (v_b - p_b(v_b, t)) &= -\frac{\partial p_s(v_s,t)}{\partial t}; \end{aligned} \tag{1}$$

[7] Jarque, Ponsatí and Sákovics [2003] study a version of the Mediated Bargaining Game with a finite set of possible prices. The set of Perfect Bayesian Equilibria there contains many strategy profiles, none of them *ex-post*. If we drop time-continuity requirement we obtain also these equilibria as Bayesian equilibria in which the agents use step functions - if an agent believes that the opponent will only bid in discrete steps, then it only makes sense to bid within the same discrete set of prices.

$\forall v, t,$ s.t. $p_s(v_s, t) = p_b(v_b, t)$;

2. $p_s(0,0) = p_b(1,0)$.

Proof. Let p^* be a ME profile, and take a $v \in [0,1]^2$. We have to verify that a best reply to a strictly type-monotone, strictly time-monotone, and differentiable strategy is also such, and then we have to derive the first order condition. We do that for the seller, a mirror argument works for the buyer. In an PEQ, it is clear that if a pair of agents with types v agree at time t, then it must be that they agree with equality, i.e.

$$p_s^*(v_s, t) = p_b^*(v_b, t), \tag{2}$$

Otherwise either one of the agents could profitably deviate against the given type of the opponent - to obtain all of the difference between the proposed prices. From equation (2), we can define by the implicit function theorem, $v_s = v_s(v_b, t)$, and we have $\frac{\partial p_s}{\partial v_s}\frac{\partial v_s}{\partial v_b} = \frac{\partial p_b}{\partial v_b}$. In Proposition 14 of the Appendix B we prove that every regular BE must be weakly type-monotone, so that $\frac{\partial v_s}{\partial v_b} \geq 0$. By assumption $\frac{\partial p_b}{\partial v_b} > 0$, therefore it must be that $\frac{\partial v_s}{\partial v_b} \geq 0$ and $\frac{\partial p_s}{\partial v_s} > 0$. Now, given p_b^*, again by (2), the seller maximizes

$$\max_{t \in [0,\infty)} e^{-t} \left(p_b^*(v_b, t) - v_s \right),$$

which immediately implies the first-order condition (FOC). It is also easy to check that the second derivative of the objective function is negative so that the FOC is indeed necessary and sufficient. The condition $p_s(0,0) = p_b(1,0)$ follows from (2) and strict type-monotonicity. □

Theorem 3. If a strategy profile p^* is a ME of MBG then the corresponding mechanism is a lottery F_p over posted prices, with a continuous density f_p, and $supp(F_p) = [0,1]$. For the converse, take a lottery F_p, cont. density f_p, $supp(F_p) = [0,1]$. Then there exists a unique p^* which is a ME of the MBG p^*, and such that F_p is the mechanism corresponding to p^*.

Proof. We first show that (1) is equivalent to PIC. We will focus on the seller, the proof for the buyer is identical. Let p^* be a differentiable strictly type-monotone profile satisfying (1). Define for each v, $\tilde{t}(v) = \min\{t \mid p_b^*(v_b, t) = p_s^*(v_s, t)\}$. From (1), by strict type-monotonicity of p^*, and applying the Implicit Function Theorem we have that $\tilde{t}$ is well-defined. Now let $\pi(v) = p_s^*(v_s, \tilde{t}(v)) = p_s^*(v_s, \tilde{t}(v))$ so that taking the derivative w.r.t. v_s we obtain $\frac{\partial \pi(v)}{\partial v_s} = \frac{\partial p_b}{\partial t}\frac{\partial \tilde{t}}{\partial v_s}$. Therefore,

$$\frac{\partial p_b}{\partial t} = \frac{1}{\frac{\partial \tilde{t}}{\partial v_s}} \frac{\partial \pi(v)}{\partial v_s}.$$

Defining $\delta(v) = e^{-\tilde{t}(v)}$, substituting this and the expression for $\frac{\partial p_b}{\partial t}$ into (1), and multiplying by $e^{-\tilde{t}(v)}$ we obtain

$$\delta(v)\frac{\partial \pi(v)}{\partial v_s} = -\frac{\partial \delta(v)}{\partial v_s}(\pi(v) - v_s).$$

This is precisely the necessary and sufficient FOC for PIRIC mechanisms given in Section 4, when π and δ are both differentiable. Since in a ME $p_s^*(0,0) = p_b^*(1,0)$, this implies that $\tilde{t}(0,1) = 0$, so that (π, δ) must be a PCE mechanism, so it is representable by some probability distribution F_p. The other properties of F_p follow immediately. For the converse, if F_p is a continuously differentiable distribution with $supp(F_p) = [0,1]$, then we can construct the equivalent representation (π, δ). Now we can do the above substitutions in the other direction, and thus construct a unique pair of strategies p^* satisfying (1), so that p^* is a ME. Thus, the solutions to (1) exist, and implement precisely all the differentiable PCE mechanisms. □

Corollary 4. The ME equilibria of MBG are PCE.

Proof. Take a ME of the MBG, $(p^*; MBG)$. Suppose there existed a dynamic bargaining game Γ and an PEQ profile $(\tilde{p}^*; \Gamma)$, dominating $(p^*; MBG)$. Now let m be the mechanism corresponding to $(p^*; MBG)$, and let $\tilde{m}$ be the mechanism corresponding to $(\tilde{p}^*; \Gamma)$. Since $(\tilde{p}^*; \Gamma)$ dominated $(p^*; MBG)$, it must be that $\tilde{m}$ dominates m, which is a contradiction by Theorem 1 and Proposition 3. □

We remark that Theorem 3 implies that the set of outcomes of ME is dense in the set of outcomes of PEQ of the MBG. The proof is quite simple. Take an PEQ of the MBG, and the associated PIRIC mechanism, which is representable as a distribution F_p over posted prices by Theorem 1. Then there exists a sequence of continuously differentiable distributions converging to F_p point-wise (on $[0,1]$), and the outcomes of the mechanisms converge point-wise (in the type space) to the outcomes under F_p. By Theorem 3, for each continuously-differentiable distribution over posted prices there is an ME of the MBG implementing that distribution. For an example of this procedure see Example 8.

In the next example we show that there is a unique ME which is linear in agents' types and the allocation is consistent with the Nash solution for all draws of types (and thus with the limit of the allocations in the Rubinstein bargaining game, as the time between the offers goes to 0). Note that in the present set-up with incomplete information, the delay occurs almost surely (i.e. except when $v_s = 0$ and $v_b = 1$). We will show in the next Section that a unique linear ME exists under more general circumstances.

Example 5. There is a unique Nash - solution consistent ME. It is given by the following type-contingent strategy profile:

$$
\begin{aligned}
p_s(v_s,t) &= \min\left\{1, v_s + \frac{e^{-t}}{2}\right\}, \\
p_b(v_b,t) &= \max\left\{0, v_b - \frac{e^{-t}}{2}\right\}.
\end{aligned}
$$

The Nash - solution prescribes $\pi(v) = \frac{v_b+v_b}{2}$. Taking a uniform distribution over posted prices in $[0,1]$ yields the mechanism $\pi(v) = \frac{v_b+vs}{2}$, $\delta(v) = \max\{v_b - v_s, 0\}$. Checking that (1) holds is a straightforward computation. It is also easy to check that no other positive density over $[0,1]$ can sustain $\pi(v) = \frac{v_b+v_b}{2}$.

We remark that our model admits an interpretation as the limit of a game of alternating moves à la Rubinstein [1982], when the length of the period goes to zero and proposals are submitted to the Mediator. Example 5 describes the unique ME profile consistent with such interpretation, since agreement at $\frac{v_b+v_b}{2}$ prevails uniquely at subgames where types have been revealed (See Binmore, Rubinstein and Wolinsky [1986]). However, note that this linear equilibrium outcome only coincides with the Nash solution when agents are risk neutral or they have the same risk aversion, see Proposition 9 in Section 6 and the subsequent comment.

Example 6. In this example we construct two ME in non-linear strategies. In the first one the strategies can be explicitly computed. The second one is symmetric, but the strategies can't be computed in closed form. Take a lottery over posted prices given by a pdf $f_p(x) = 2x, x \in [0,1]$. Note that f_p is differentiable and strictly positive, so that the corresponding strategies of the MBG will satisfy all the conditions for a ME. To construct the strategies proceed as follows. First,

$$
\delta(v) = \int_{v_s}^{v_b} f_p(\tau)d\tau = v_b^2 - v_s^2, \text{ and}
$$

$$
\pi(v) = E_{f_p}[p \mid p \in [v_s, v_b]] = \frac{1}{v_b^2 - v_s^2}\int_{v_s}^{v_b} \tau f_p(\tau)d\tau = \frac{2}{3}\frac{v_b^2 + v_s v_b + v_s^2}{v_b + v_s}.
$$

Since $\delta(v) = e^{-t}$, where t is the time of agreement between types v_s and v_b, we get $\tilde{v}_b(v_s,t) = \sqrt{e^{-t} + v_s^2}$, where $\tilde{v}_b(v_s,t)$ is the type of buyer who agrees with the seller v_s at time t. Noting that $p_s(v_s,t) = \pi(v_s, \tilde{v}_b(v_s,t))$ we obtain

$$
p_s(v_s,t) = \frac{2\left(e^{-t} + 2v_s^2 + v_s\sqrt{e^{-t} + v_s^2}\right)}{3\left(v_s + \sqrt{e^{-t} + v_s^2}\right)}.
$$

Similarly, we could compute the strategy of the buyer.

For the second example consider $f_p(x) = 6x(1-x)$. Then $\pi(v) = \frac{2(v_b^3 - v_s^3) - \frac{3}{2}(v_b^4 - v_s^4)}{3(v_b^2 - v_s^2) - 2(v_b^3 - v_s^3)}$ and $\tau(v) = -\ln \delta(v)$, where $\delta(v) = F(v_b) - F(v_s) = (3v_b^2 - 2v_b^3) - (3v_s^2 - 2v_s^3)$. Thus, the strategy of the buyer is $p_b(v_s, t) = \pi(v_b, \chi(v_b, t))$ where $\chi(v_b, t))$ solves $3v_b^2 - 2v_b^3 - e^{-t} = 3\chi^2(v_b, t) - 2\chi^3(v_b, t)$, and similarly for the seller.

Next, we provide a simple example of an PEQ which is not a ME.

Example 7. Let $p^* \in [0, 1]$, and consider the following strategies of the traders. The seller's types $v_s \leq p^*$ commit to always demanding p^*, and the types $v_s > p^*$ commit to always demanding 1. Similarly, the buyer's types $v_b \geq p^*$ always bid p^*, and $v_b < p^*$ always bid 0. It is trivial to check that this is an PEQ of the MBG, and it is clearly not a ME. The direct-revelation mechanism corresponding to this PEQ is a degenerate distribution F_p (by virtue of Theorem 1) with point mass at $p^* \in [0, 1]$.

Using this logic, and the representation of Theorem 1, the reader can construct more contrived examples at will. That is, take some distribution F_p which is not continuous w.r.t. the Lebesque measure, and there exists an PEQ (which is not a ME) of the MBG, such that the direct mechanism corresponding to that PEQ is the given F_p.

Finally, we present a standard example of welfare analysis in terms of *ex-ante* constrained efficiency. We again stress that PCE is necessary for *ex-ante* constrained efficiency. Thus, it is enough to look for optimal mechanisms within the class of probability distributions over posted prices. Moreover, under risk-neutrality, the *ex-ante* optimal mechanism is a deterministic posted price (i.e. a point-mass at the *ex-ante* optimal posted price). By the previous example, the corresponding PEQ is not a ME. (In contrast, under risk aversion the *ex-ante* optimal PEQ is generically a ME, see Example 11 of Section 6, and Čopič and Ponsatí [2005].)

Example 8. Let v_b and v_s be iid, uniform on $[0, 1]$. For simplicity we find the *ex-ante* constrained-efficient mechanism that maximizes the sum of expected utilities. In this case it is quite obvious that the only candidate is by symmetry a posted price $p^* = \frac{1}{2}$ (i.e. a degenerate distribution over posted prices with a point-mass at $\frac{1}{2}$). The welfare under this mechanism is

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 (\frac{1}{2} - v_s) + (v_b - \frac{1}{2}) dv_b dv_s = \frac{1}{8}.$$

From the previous Example we know that there is an PEQ of MBG corresponding to this mechanism, but this PEQ is not an ME. On the other hand, it is straightforward that for each continuously differentiable F_p with full support, there is an ME of MBG which implements that F_p. Therefore, there exists a sequence of ME, approximating the outcome under $p^* = \frac{1}{2}$ (point-wise in the type space) - take for instance $f^n = k_n x^n (1-x)^n, n = 1, 2, ...,$ and k_n is chosen so that f_n integrates to 1. Thus, the outcome under p^* is a limit point of the set of ME outcomes. The welfare under the linear ME is $\frac{1}{12}$, and the welfare under the ME corresponding to $f_p(x) = 6(1-x)x$ is $\frac{1}{10}$. For comparison, in this same environment, Cramton [1992] computes a symmetric stationary separating PBE where expected benefits equal to $\frac{3}{32}$, so that the *ex-ante* ranking of welfare in these equilibria is $\frac{1}{8} > \frac{1}{10} > \frac{3}{32} > \frac{1}{12}$ (the optimal robust PEQ $\succ$ symmetric non-linear ME of Example 6 $\succ$ Cramton's PBE $\succ$ the linear ME). Note that the PBE in Cramton[1992] is not an PEQ (thus it is not robust). Also note that when agents are risk-averse, the linear ME is *ex-ante* more efficient than the most efficient posted price - see the continuation of this example, Example 11 of Section 6.

6 Risk aversion and unequal discount rates.

In this Section, we discuss the MBG in a slightly richer model. The agents' static preferences display constant relative-risk aversion (CRRA), i.e. $u_s(v_s, p) = (p - v_s)^{\gamma_s}, u_b(v_b, p) = (v_b - p)^{\gamma_b}$, where $\gamma_i \in (0, 1], i = s, b$. The agents are allowed to discount the future differently, so that time preference of i is given by $\rho_i \geq 1$ - agent i discounts according to $e^{-\rho_i t}$, $i = s, b$. The restriction that $\rho_i \geq 1$ is without loss of generality since all that matters are relative rates of discounting. Parameters γ and ρ are common knowledge. We will show that risk-aversion and time preference act as substitutes, so that behaviorally, an agent that is more impatient *acts as if he were more risk-averse*. In particular, this is true in a static direct mechanism, so that the mechanism has to be adjusted for risk-aversion *and impatience* - even though the game is static. The point is that it matters that a direct mechanism is a reduced form of a dynamic game.

One could consider a richer model, where each agent has some concave utility function and some discounting criterion, and the results would not change substantially. We limit ourselves to the present setup mostly for the sake of tractability and also because the present case exhausts the environments where MBG admits ME in linear strategies. It is worth noting that in a dynamic game, for agents to display preferences that are consistent in the inter-temporal sense, we have to restrict the agents instantaneous utility functions to display constant relative-risk aversion (CRRA) (See Fishburn and Rubinstein [1982]). Nonetheless, redoing the present exercise under other behavioral assumptions may be interesting.

We first derive the FOC for a ME in this environment (the argument is identical to the argument in the Proof of Proposition 2 above). So let p^* be a ME profile.

Now, given p_b^*, the seller considers the problem

$$\max_{t\in[0,\infty)} e^{-\rho_s t}\left(p_b^*(v_b,t)-v_s\right)^{\gamma_s},$$

which yields the first order condition (similarly for the buyer)

$$(p_b^*(v_b,t)-v_s)=\frac{\gamma_s}{\rho_s}\frac{\partial p_b^*(v_b,t)}{\partial t}. \tag{3}$$

Observe that it is impossible to distinguish the first order condition for agents that are risk averse from the first order condition for impatient agents. In particular, in the direct mechanism, we need to consider $\gamma_i' = \frac{\gamma_i}{\rho_i}$ as the risk-aversion parameter of agent i.

Consider first the case when $\rho_s = \rho_b = \rho$, so that $\gamma_i' = \frac{\gamma_i}{\rho}$, $\delta_v = e^{-\rho\tilde{t}(v)}$, and $\pi(v) = p_b^*(v_b, \tilde{t}(v)) = p_s^*(v_s, \tilde{t}(v))$, where again $\tilde{t}(v) = \min\{t \mid p_b^*(v_b,t) = p_s^*(v_s,t)$, to obtain

$$(\pi(v)-v_s)\frac{\partial\delta(v)}{\partial v_s} = -\gamma_s'\delta(v)\frac{\partial\pi(v)}{\partial v_s}. \tag{4}$$

By differentiability of the profile p^* both δ and π are differentiable, and (4) is precisely the PIRIC condition for differentiable mechanisms when γ' are the risk aversion parameters, which is easy to check along the lines of Section 4.

When $\gamma_i' \neq 1$ for at least one i there is no representation of PIRIC mechanisms in terms of distributions over posted prices as in Theorem 1. Still, for each mechanism $m = (\delta, \pi)$, satisfying (4), we can construct by the above substitutions exactly one strategy profile p^* satisfying the necessary and sufficient conditions (3) for a ME. Thus, the analog to Theorem 3 holds. For a more detailed treatment of PIRIC mechanisms under risk aversion see Čopič and Ponsati [2005], where we also prove that the mechanisms described by the equation (4) are PCE. [8]

Since $\delta(v) = e^{-\rho\tilde{t}(v)}$, each trader perceives the deterministic trade at price $\pi(v)$ and time $\tilde{t}(v)$ exactly the same as instantaneous trade at price $\pi(v)$ with probability $\delta(v)$. The price is distorted due to risk-sharing, and the probability may be affected by impatience as well. Notice that we could also reparametrize time to $\tau = \rho t$, and under this new time-scale there would be no distortion of perceived probability (i.e., agents getting older faster or being more impatient is formally equivalent).

Similarly, when $\rho_s \neq \rho_b$ what matters is the $\gamma_i' = \frac{\gamma_i}{\rho_i}, i = s, b$, and Equation 4 still describes the PIRIC condition for the direct mechanisms. Therefore, the difference

[8] The main problem is that in a non-linear environment there are PIRIC mechanisms that cannot be represented as binary lotteries (since an agent is no longer indifferent between the lottery and its mean, both on and off the equilibrium path). In Čopič and Ponsati [2005] we prove that the mechanisms that are binary lotteries are PCE.

of the relative impatience also has an effect on pricing, as well as on the probability of trade. Again, even in the static set-up of direct mechanisms we have to take into account the impatience, and not only the risk-aversion of the agents. Behaviorally, more impatient agents act as if they were more risk-averse. Naturally, now there doesn't exist a re-scaling of time units that would work for both traders. See also Example 10 at the end of this Section.

For the rest of this Section we limit ourselves to the unique mechanism (and ME of the MBG) where pricing is linear in agents' types. We remark that in environments where agents' risk attitudes are not CRRA or they don't discount the future exponentially, no linear pricing mechanism exists (see Čopič and Ponsatí [2005]).

Proposition 9. Given the environment described by (γ, ρ), there exists a unique solution (δ, π) to (4) such that $\delta(v)$ is linear in v and $\delta(0,1) = 1$. More precisely,

$$\pi(v) = \frac{\sqrt{\gamma_s'}}{\sqrt{\gamma_s'} + \sqrt{\gamma_b'}} v_b + \frac{\sqrt{\gamma_b'}}{\sqrt{\gamma_s'} + \sqrt{\gamma_b'}} v_s, \delta(v) = (v_b - v_s)^{\sqrt{\gamma_s' \gamma_b'}}, v_b \geq v_s. \qquad (5)$$

Proof. Let $\pi(v) = \alpha v_s + (1 - \alpha) v_b$, and insert this into (4). This gives

$$\frac{\partial \log \delta(v)}{\partial v_s} = -\frac{\gamma_s' \alpha}{1 - \alpha} \frac{1}{v_b - v_s},$$

$$\frac{\partial \log \delta(v)}{\partial v_b} = \frac{\gamma_b'(1 - \alpha)}{\alpha} \frac{1}{v_b - v_s}.$$

By integrating the first equation we obtain $\log \delta(v) = \frac{\gamma_s' \alpha}{1-\alpha} \log(v_b - v_s) + K_s(v_b)$, and from the second we obtain $\log \delta(v) = \frac{\gamma_b'(1-\alpha)}{\alpha} \log(v_b - v_s) + K_b(v_s)$, where $K_s(v_b)$ and $K_b(v_s)$ are integration constants. But then it must be that $K_s = K_b = const.$ (determined from $\delta(0,1) = 1$) and $\frac{\gamma_b'(1-\alpha)}{\alpha} = \frac{\gamma_s' \alpha}{1-\alpha}$. Therefore α is uniquely determined. □

Observe that pricing under the linear mechanism is different from the Nash-solution pricing, which is $\frac{\gamma_s'}{\gamma_s'+\gamma_b'} v_b + \frac{\gamma_b'}{\gamma_s'+\gamma_b'} v_s$, and is still attainable as the limit of SPNE of the Rubinstein alternating offers game, when v_s and v_b are known. This difference is not surprising since the contraction independence property that is required in the (generalized) Nash solutions, and implied by equilibrium conditions in the Rubinstein game, is clearly not equivalent to the incentive constraints. However, it is remarkable that risk-sharing goes in the same direction for both allocations: the more risk averse agent obtains less surplus. Note also that in both models (the present one and the Rubinstein alternating-offers model) the risk-sharing and impatience have an effect on pricing which goes in the same direction: the more impatient agent gets less, and the more risk-averse agent gets less. We conclude this Section with two examples.

Example 10. Let $\frac{\gamma_s}{\rho_s} = \gamma'_s = \frac{\gamma_b}{\rho_b} = \gamma'_b = \gamma'$, so that the unique linear mechanism is given by $\pi(v) = \frac{1}{2}(v_1 + v_2), \delta(v) = (v_2 - v_1)^{\gamma'}$. The strategies of the agents in the appropriate ME of the MBG are given by

$$p_s^*(v_s, t) = v_s + \frac{e^{-\gamma' t}}{2}, p_b^*(v_b, t) = v_b - \frac{e^{-\gamma' t}}{2}.$$

Thus, there is a two parametric family of environments where the utility outcome is invariant, and it is the same regardless of whether each agent is impatient or risk-averse. We remark that while in the static direct mechanism sense we have to adjust for dynamic time preference, in the dynamic setting we have to adjust for the static risk-aversion, even though at each time the outcome is deterministic.

In the last example we show that when agents are risk-averse the *ex-ante* optimal (under a utilitarian social welfare function) PEQ of the MBG is a ME. Computing the *ex-ante* optimal mechanism is a bit complicated (and can in general only be done numerically), and we refer an interested reader to Čopič and Ponsatí [2005].

Example 11. Let $\gamma_s = \gamma_b = \gamma, \gamma \in (0, 1]$ and $\rho_s = \rho_b = 1$, and as before, let the social welfare be given by $u_s + u_b$. Also, let v_b and v_s be iid, uniform on $[0, 1]$. Then, by symmetry, the most *ex-ante* efficient posted price is $p^* = \frac{1}{2}$. The *ex-ante* social welfare under p^*, as a function of risk-aversion γ is

$$W^p(\gamma) = \frac{1}{2(\gamma + 1)} \left(\frac{1}{2}\right)^{\gamma+1},$$

and the social welfare under the linear ME is

$$\frac{1}{2(\gamma + 1)(2\gamma + 1)} \left(\frac{1}{2}\right)^{\gamma}.$$

These two expressions are equal when $\gamma = \frac{1}{2}$. For more risk-averse traders (i.e. $\gamma < \frac{1}{2}$) social gains are higher under the linear ME. In fact, the linear ME approaches *ex-post* efficiency as the traders' risk aversion goes to infinity.

7 Separating PBE are ME

We now show that every separating Perfect Bayesian equilibrium (PBE) of the MBG must be a ME. We note that in the MBG, the off-equilibrium deviations are unobservable so that the set of outcomes of PBE and the set of outcomes of BE coincide. In Appendix B we show that all BE must be weakly type monotone (see Proposition **??**).

Beliefs. Recall from Section 2 that it is common knowledge that types are drawn from some distribution with support $[0,1]^2$. Presently we must assume that the specific pdf G is common knowledge. Agent i updates her beliefs over the distribution of the opponent's types over time. As described in Section 4, the histories depend only on t. Thus, given a strategy profile p, the beliefs of a player about the opponent are updated only as a function of time. We denote by $G_j(v_j|v_i,t;p)$ the distribution of the belief of agent i of type v_i about agent j at time t, conditional on no agreement until time t. By $g_j(v_j|v_i,t;p)$ we denote the density of G_j, whenever it exists. Finally, we denote by $H_j(v_i,t;p)$ the mass of types of player j with whom agent i has agreed with by time t. We will economize the notation and omit parameters v_i and p whenever that is unambiguous. Note that if the strategies of both players are differentiable with respect to both parameters, and these partial derivatives are non-zero, the beliefs will be differentiable with respect to time.

Bayes and Perfect Bayes-Nash equilibrium. Denote by $EU_i(v_i;p,G)$ the expected payoff of player i of type v_i, when agents play according to strategy profile p and types are distributed according to G. Let $G_j(v_i)$ denote the conditional distribution of j's types. Thus,

$$EU_i(v_i;p,G) = \int_0^1 u_i(\bar{p}i(p,v_i,v_j),v_i)\, e^{-\tau(p,v_i,v_j)} dG_j(v_i),$$

or alternatively

$$EU_i(v_i;p,G) = \int_{t\in[0,\infty)} u_i(p_i(v_i,t),v_i)\, e^{-t} dH_j(v_i,t),$$

where both of these integrals have to be understood as Lebesgue integrals.

Denote by Π_i the set of strategies for player i, and by $\Pi = \Pi_s \times \Pi_b$ the set of strategy profiles. A strategy profile $p = (p_i, p_j) \in \Pi$ constitutes a *Bayes Nash equilibrium* if and only if

$$EU_i(v_i;p,G) \geq EU_i(v_i;p'_i,p_j,G),\ \forall p'_i \in \Pi_i,$$

for all $v_i \in [0,1]$, $i = s,b$, $j \neq i$.

A careful definition of the PBE in our setting requires specifying agents' expected utility in every subgame, which in our setup means at every time t. Let $EU_i(v_i,t;p,G)$ denote the expected payoff to player i of type v_i in the subgame starting at t, when agents play strategies p (note that p, v_i, and t also specify the history observed by agent i):

$$EU_i(v_i, t; p, G) = \int_{\tau \in [t, \infty)} u_i(p_i(v_i, \tau), v_i) e^{-\tau} dH_j(v_i, \tau)$$

A strategy profile $p \in \Pi$ constitutes a *Perfect Bayesian equilibrium* if

$$\begin{aligned} EU_i(v_i, t; p, G) &\geq EU_i(v_i, t; p'_i, p_j, G), \\ \forall p'_i &\in \Pi_i \text{ s.t. } p'_i(v_i, t') = p_i(v_i, t') \text{ for all } t' \leq t, \end{aligned}$$

for all $t \geq 0$, for all $v_i \in [0,1]$, $i = s, b$, $j \neq i$. As we noted earlier, BE and PBE are outcome-equivalent in the MBG.

We impose the following regularity condition and restrict attention to BE in regular strategies. Note that a BE in regular strategies is a BE of the MBG which we show in Corollary 19, in the Appendix B.

R3 We say that a strategy is *regular* if $\frac{\partial p_i(v_i,t)}{\partial v_i}$ is continuous $\forall t \in [0, \infty)$ and $\lim_{t \to \infty} p_i(v_i, t)$ is a left-continuous function of v_i, for all $v_i \in [0,1]$.

SEP We say that a regular strategy is *separating* if $\frac{\partial p_i(v_i,t)}{\partial v_i} \neq 0$, $\forall\ t \in [0, \infty)$ and $\forall v_i \in (0,1)$.

The regularity condition R3 imposes a pattern of behavior that rules out dramatic changes when types change only marginally, which is a natural requirement since types and dates take values in a continuum. The second part of R3 is roughly an indifference breaking rule: if an agent of some type is at the horizon indifferent between two concessions to the opponent, she will concede more. In Lemma 15 in the Appendix B we show that this condition is enough to assure the continuity of the demands with respect to types at the time horizon and that in a regular equilibrium the agents' bids asymptotically approach the reservation values.

With the main theorem of this Section we wrap up our paper.

Theorem 12. All regular and separating PBE (and thus BE) of the MBG are ME.

The sketch of the proof goes roughly as follows. First, we show that a differential first order condition for a regular BE is well defined. Then we show that the strategies resulting from this first order condition must be belief independent so that a BE is an PEQ. The intuition behind this is that a separating equilibrium is fully revealing, i.e. for each proposal and each date the seller will know exactly the valuation of the opponent with whom she agrees at that proposal and date. Thus, once the agreement occurs the agents know each other's types, and since this is common knowledge *ex-ante*, they must play best-replies against the strategy of each type of the other player. For details see Appendix B.

Appendix A.

Proof of Theorem 1.

We show that a mechanism is PIRIC if and only if it can be represented as a distribution, not necessarily with mass 1, over posted prices. The theorem then follows, since only distributions with mass 1, i.e. probability distributions over posted prices can be PCE. The proofs presented here are a bit denser, for a more comprehensive version see Čopič and Ponsatí [2005].

First, take a distribution F_p over posted prices. The simple form of this mechanism, $m = (\pi, \delta)$, is given by $\delta(v) = \max\{F_p(v_b) - F_p(v_s), 0\}$, $\pi(v) = E_{F_p}[\omega \mid \omega \in [v_s, v_b]]$. Clearly, m is PIRIC, since distributions over posted prices are PIRIC (the draw of the price is independent of traders' reports so that reporting truthfully is a dominant strategy, PIR is enforced by definition).

To prove the converse take a mechanism $m = (\pi, \delta)$ satisfying PIRIC. We have to show that PIRIC implies that π and δ can be represented (as claimed) by some F_p. It is enough to show that there exists a monotonic F_p such that $\delta(v) = \max\{F_p(v_b) - F_p(v_s), 0\}$. From PIR and PIC, it easily follows (see for instance Theorem 1 in Hagerty and Rogerson [1987], our δ corresponds to their p) that δ has to satisfy

$$\delta(v_s, v_s) = \frac{1}{v_b - v_s} \int_{v_s}^{v_b} \delta(\tau, v_b) + \delta(v_s, \tau) d\tau, \forall (v_s, v_b) \in [0,1]^2, \tag{6}$$

and $\delta(v_s, v_b)$ is bounded, increasing in v_s, decreasing in v_b, and non-negative for $(v_s, v_b) \in [0,1]^2$. The proof now follows from the following theorem. This Theorem also appears in Čopič [2005] and Čopič and Ponsatí [2005] with the same proof. Here we include the proof for reader's convenience.

Theorem 13. Let a function $\delta(v_s, v_b)$ be bounded, increasing in v_s, decreasing in v_b, and non-negative, and satisfy (6) for all $(v_s, v_b) \in [0,1]^2$. Then there exists a monotonically increasing function $\tilde{\delta} : [0,1] \to [0,1]$, such that $\delta(v_s, v_b) = \tilde{\delta}(v_b) - \tilde{\delta}(v_s), \forall v_b \geq v_s$, and $\delta(v_s, v_b) = 0 \forall v_b < v_s$.

Proof. We prove the claim in a few steps. To understand the logic it is best to think of $\tilde{\delta}(.)$ as a distribution, which induces a measure $\tilde{\mu}$. We know that $\tilde{\delta}(.)$ is continuous if and only if $\tilde{\mu}$ is continuous with respect to the Lebesgue measure, and that in general, $\tilde{\mu}$ can be decomposed into $\tilde{\mu}_l + \tilde{\mu}_o$, where $\tilde{\mu}_l$ is continuous w.r.t. the Lebesgue measure, and $\tilde{\mu}_o$ is orthogonal w.r.t. the Lebesgue measure (i.e. the jumps in $\tilde{\delta}(.)$). Moreover, $\tilde{\delta}(.)$ is continuous if and only if $\delta(v_s, v_b)$ is continuous in each of the two dimensions (i.e. v_s and v_b). In Case 1, we treat the problem when $\delta(v_s, v_b)$ is continuous. In Case 2, we treat the general problem when $\delta(v_s, v_b)$ can be discontinuous. If $\delta(v_s, v_b)$ is not continuous in each dimension the set of discontinuities of $\delta(v_s, v_b)$ could be very complex. Then, the fact that $\delta(v_s, v_b)$ can be represented by $\tilde{\delta}(.)$ implies that the discontinuities of $\delta(v_s, v_b)$ have a very specific structure. That is, if for a fixed v_s, $\delta(v_s, \tau)$ is discontinuous at some $\bar{\tau} \geq v_s$, then $\delta(v_s', \tau)$ is discontinuous at $\bar{\tau}$ for all $v_1' < \bar{\tau}$ (Step 2.1.), and $\delta(\tau, v_b)$ is discontinuous at $\bar{\tau}$ for all $v_b > \bar{\tau}$ (Step 2.2.).

Case 1. Let $\delta(v_s, \tau)$ and $\delta(\tau, v_b)$ be continuous in τ, for every $(v_s, v_b) \in [0, 1]^2$.

We define $\phi(v_s, v_b, t) = \delta(v_s, t) + \delta(t, v_b) - \delta(v_s, v_b)$, and we prove that $\phi(v_s, v_b, t) = 0, \forall t \in [v_s, v_b]$. Note that ϕ is continuous in each of its arguments, in particular it is continuous in t. We proceed as follows. In Step 1.1. we show that there exists a $\bar{t} \in (v_s, v_b)$ s.t. $\phi(v_s, v_b, \bar{t}) = 0$. In Step 1.2. we show that $\frac{\partial \phi(v_s, v_b, t)}{\partial t} = 0$ everywhere by showing that the derivative of $\phi(v_s, v_b, t)$ w.r.t. t from the left is equal to that derivative from the right everywhere (and both are equal to 0). From the definition of ϕ it is clear that its derivative from the left w.r.t. t will be equal to 0 if and only if the derivative from the left of $f(v_s, t)$ w.r.t. t is equal the derivative from the left of $f(t, v_b)$ w.r.t. t, which is precisely what we show in Step 1.2. Similarly for the derivative from the right. Thus, ϕ is differentiable, its derivative is 0, and it is equal to 0 at some point by Step 1.1. - then ϕ must be equal to 0 everywhere. While Step 1.1. is straightforward, Step 1.2. involves some calculus.

Step 1.1. There exists a $\bar{t} \in (v_s, v_b)$ s.t. $\phi(v_s, v_b, \bar{t}) = 0$.

Proof. Now (6) can be written as

$$0 = \frac{1}{v_b - v_s} \int_{v_s}^{v_b} \phi(v_s, v_b, \tau) d\tau.$$

By the mean value theorem (MVT), there exists a $\bar{t} \in (v_s, v_b)$, s.t. $\frac{1}{v_b - v_s} \int_{v_s}^{v_b} \phi(v_s, v_b, \tau) d\tau = \phi(v_s, v_b, \bar{t})$, which concludes the proof of Step 1.1.

Step 1.2. $\phi(v_s, v_b, t)$ is differentiable in t and $\frac{\partial \phi(v_s, v_b, t)}{\partial t} = 0$, for all $t \in (v_s, v_b)$.

Proof. Denote by

$$\frac{\partial^+ \delta(v_s, t)}{\partial t} = \lim_{\epsilon \to 0, \epsilon > 0} \frac{\delta(v_s, t + \epsilon) - \delta(v_s, t)}{\epsilon}$$

the derivative from the right of $\delta(v_s, t)$ w.r.t. t. Similarly, let $\frac{\partial^- \delta(v_s, t)}{\partial t}$ denote the derivative from the left. We will show that for every $t \in (v_s, v_b)$,

$$\frac{\partial^+ \phi(v_s, v_b, t)}{\partial t} = \frac{\partial^- \phi(v_s, v_b, t)}{\partial t} = 0.$$

We will do that by showing that $\frac{\partial^+ \delta(v_s,t)}{\partial t} = -\frac{\partial^+ \delta(t,v_b)}{\partial t}$ and $\frac{\partial^- \delta(v_s,t)}{\partial t} = -\frac{\partial^- \delta(t,v_b)}{\partial t}$, for all $t \in (v_s, v_b)$. Note that the left and the right-derivatives of $\delta(v_s, t)$ and $\delta(t, v_b)$ w.r.t. t exist for all t since δ is continuous and monotonic.

We first show that

$$\frac{\partial^+ \delta(v_s, t)}{\partial t} = \frac{\partial \delta^+(v_s', t)}{\partial t}, \forall v_s', v_s < t. \tag{7}$$

To see this, we write by definition,

$$\frac{\partial^+ \delta(v_s, t)}{\partial t} = \lim_{\epsilon \to 0, \epsilon > 0} \frac{1}{\epsilon} \left(\delta(v_s, t + \epsilon) - \delta(v_1, t) \right).$$

We now use (6) and compute

$$\delta(v_s, t+\epsilon) - \delta(v_s, t) = \int_{v_s}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau - \int_{v_s}^{t} \frac{\delta(v_s,\tau) + \delta(\tau, t)}{t - v_s} d\tau$$

$$= \int_{t}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau + \int_{v_s}^{t} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} - \frac{\delta(v_s,\tau) + \delta(\tau, t)}{t - v_s} d\tau$$

$$= \int_{t}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau + \int_{v_s}^{t} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} - \frac{\delta(v_s,\tau) + \delta(\tau, t)}{t - v_s} d\tau$$

$$= \int_{t}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau + \int_{v_s}^{t} \frac{-\epsilon(\delta(v_s,\tau) + \delta(\tau, t))}{(t+\epsilon - v_s)(t - v_s)} + \frac{\delta(\tau, t+\epsilon) - \delta(\tau, t)}{t+\epsilon - v_s} d\tau$$

$$= \int_{t}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau - \frac{\epsilon \delta(v_s, t)}{t+\epsilon - v_s} + \int_{v_s}^{t} \frac{\delta(\tau, t+\epsilon) - \delta(\tau, t)}{t+\epsilon - v_s} d\tau$$

From this last expression we can see that $\lim_{\epsilon \to 0, \epsilon > 0} \frac{1}{\epsilon} \left(\delta(v_s, t+\epsilon) - \delta(v_s, t) \right) = \frac{1}{t+\epsilon - v_s} \int_{v_s}^{t} \frac{\partial^+ \delta(\tau, t)}{\partial v_b} d\tau$, since

$$\lim_{\epsilon \to 0, \epsilon > 0} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} \frac{\delta(v_s,\tau) + \delta(\tau, t+\epsilon)}{t+\epsilon - v_s} d\tau - \frac{\delta(v_s, t)}{t+\epsilon - v_s} = 0,$$

by the MVT.

This implies that indeed (7) holds. Similarly, we obtain $\frac{\partial^+ \delta(t,v_b)}{\partial t} = \frac{\partial \delta^+(t,v_b')}{\partial t}, \forall v_b', v_b > t$.

Now take a monotonic sequence $\epsilon_n, n = 1, ..., \infty$, s.t. $\lim_{n\to\infty} \epsilon_n = 0$, and let $v_{b,n}' =$

$t+\epsilon_n$. By above, we know that for each n,

$$\lim_{l\to\infty, l\geq n} \frac{\delta(t+\epsilon_l, v'_{b,n}) - \delta(t, v'_{b,n})}{\epsilon_l} = \frac{\partial^+\delta(t, v'_{b,n})}{\partial t} = \frac{\partial^+\delta(t, v_b)}{\partial t}.$$

Then, by the Cauchy diagonalization theorem, it is also true that

$$\lim_{n\to\infty} \frac{\delta(t+\epsilon_n, v'_{b,n}) - \delta(t, v'_{b,n})}{\epsilon_n} = \frac{\partial^+\delta(t, v_b)}{\partial t}. \tag{8}$$

Next, since $\delta(t,t) = 0$, and also applying (7), we have for ϵ_n sufficiently small (i.e. n large enough),

$$\delta(t, v'_{b,n}) = \delta(t, t+\epsilon_n) = \delta(t,t) + \frac{\partial^+\delta(t,t)}{\partial v_b}\epsilon_n + O(\epsilon^2) = \frac{\partial^+\delta(v_s, t)}{\partial v_b}\epsilon_n + O(\epsilon_n^2).$$

Note that $\frac{\partial^+\delta(t,t)}{\partial v_b}$ is understood as $\lim_{v_b\to t, v_b>t} \frac{\partial^+\delta(t,v_b)}{\partial v_b}$. We insert this into (8), also noting that $\delta(t+\epsilon_n, v'_{b,n}) = \delta(t+\epsilon_n, t+\epsilon_n) = 0$, to obtain

$$\frac{\partial^+\delta(t, v_b)}{\partial t} = \lim_{n\to\infty} \frac{\delta(t+\epsilon_n, v'_{b,n}) - \delta(t, v'_{b,n})}{\epsilon_n} = \lim_{n\to\infty} \frac{-\frac{\partial^+\delta(v_s,t)}{\partial v_b}\epsilon_n + O(\epsilon_n^2)}{\epsilon_n} = -\frac{\partial^+\delta(v_s, t)}{\partial v_b}.$$

Thus we have shown that at every $t \in (v_s, v_b)$, $\frac{\partial^+\delta(t,v_b)}{\partial t} = -\frac{\partial^+\delta(v_s,t)}{\partial v_b}$, which implies that $\frac{\partial^+\phi(v_s,v_b,t)}{\partial t}$ exists and is equal to 0. Similarly, we show that $\frac{\partial^-\phi(v_s,v_b,t)}{\partial t}$ exists and is equal to 0, which proves that $\phi(v_b, v_b, t)$ is differentiable w.r.t. t. This concludes the proof of Step 1.2, and thus the proof of Case 1.

Case 2. The general case. We will complete the proof of this case by showing that the $\delta(v_s, v_b)$ can only be discontinuous in a way which still admits a representation by some $\tilde{\delta}(.)$. We proceed in 2 steps, both involve applying the Monotone Convergence Theorem (MCT), and some tedious calculus. The outline of these steps is described in the introductory outline of the proof.

Step 2.1. If $\exists v_s \in [0,1]$, and $\bar{\tau} > v_s$ s.t. $\delta(v_s, \bar{\tau}+) - \delta(v_s, \bar{\tau}-) = \Delta_s(v_s, \bar{\tau}) > 0$, then $\delta(v'_b, \bar{\tau}+) - \delta(v'_s, \bar{\tau}-) = \Delta_s(v_s, \bar{\tau}) > 0$, $\forall v'_s < \bar{\tau}$.

Proof. We write

$$\delta(v_s, \bar{\tau}+) = \lim_{\epsilon\to 0} \frac{1}{\bar{\tau}+\epsilon - v_s} \int_{v_s}^{\bar{\tau}+\epsilon} \delta(v_s, \tau) + \delta(\tau, \bar{\tau}+\epsilon) d\tau,$$

$$\delta(v_s, \bar{\tau}-) = \lim_{\epsilon \to 0} \frac{1}{\bar{\tau} - \epsilon - v_s} \int_{v_s}^{\bar{\tau}-\epsilon} \delta(v_s, \tau) + \delta(\tau, \bar{\tau} - \epsilon) d\tau,$$

and since

$$\lim_{\epsilon \to 0} \frac{1}{\bar{\tau} + \epsilon - v_s} = \lim_{\epsilon \to 0} \frac{1}{\bar{\tau} - \epsilon - v_s} = \frac{1}{\bar{\tau} - v_s},$$

we have

$$\Delta_s(v_s, \bar{\tau}) = \frac{1}{\bar{\tau} - v_s} \left[\lim_{\epsilon \to 0} \int_{\bar{\tau}-\epsilon}^{\bar{\tau}+\epsilon} \delta(v_s, \tau) d\tau + \lim_{\epsilon \to 0} \int_{v_s}^{\bar{\tau}+\epsilon} \delta(\tau, \bar{\tau} + \epsilon) d\tau - \int_{v_s}^{\bar{\tau}-\epsilon} \delta(\tau, \bar{\tau} - \epsilon) d\tau \right]. \tag{9}$$

Now

$$\lim_{\epsilon \to 0} \int_{\bar{\tau}-\epsilon}^{\bar{\tau}+\epsilon} \delta(v_1, \tau) d\tau = \lim_{\epsilon \to 0} \int_{v_s}^{1} 1_{(\bar{\tau}-\epsilon, \bar{\tau}+\epsilon)} \delta(v_s, \tau) d\tau = 0,$$

by the (MCT). Similarly, we apply the (MCT) to the other part of (9), so that

$$\lim_{\epsilon \to 0} \int_{v_s}^{\bar{\tau}+\epsilon} \delta(\tau, \bar{\tau}+\epsilon) d\tau - \int_{v_s}^{\bar{\tau}-\epsilon} \delta(\tau, \bar{\tau}-\epsilon) d\tau = \lim_{\epsilon \to 0} \int_{v_s}^{1} 1_{(v_s, \bar{\tau}+\epsilon)} \delta(\tau, \bar{\tau}+\epsilon) - 1_{(v_s, \bar{\tau}+\epsilon)} \delta(\tau, \bar{\tau}+\epsilon) d\tau$$

$$= \int_{[v_s, \bar{\tau})} \delta(\tau, \bar{\tau}+) - \delta(\tau, \bar{\tau}-) d\tau.$$

Therefore,

$$\Delta_s(v_s, \bar{\tau}) = \frac{1}{\bar{\tau} - v_s} \int_{[v_s, \bar{\tau})} \delta(\tau, \bar{\tau}+) - \delta(\tau, \bar{\tau}-) d\tau = \frac{1}{\bar{\tau} - v_s} \int_{[v_s, \bar{\tau})} \Delta_1(\tau, \bar{\tau}) d\tau. \tag{10}$$

The claim now follows for $v_s < \bar{v}_s < \bar{\tau}$. This concludes the proof of Step 2.1.

Step 2.2. If $\exists v_s \in [0, 1]$, and $\bar{\tau} > v_s$ s.t. $\delta(v_s, \bar{\tau}+) - \delta(v_s, \bar{\tau}-) = \Delta > 0$, then $\exists v_b > \bar{\tau}$ s.t. $\delta(\bar{\tau}-, v_b) - \delta(\bar{\tau}+, v_b) = \Delta$.

Proof. First observe that since $\delta(0, \tau)$ is monotonic, there exists a $\bar{v}_b$ such that $\delta(0, \tau)$ is continuous for $\tau \in (\bar{\tau}, \bar{v}_b]$. By Step 2, $\delta(v_s, \tau)$ is continuous for $\tau \in (\bar{\tau}, \bar{v}_2], \forall v_s < \bar{v}_b$. We can proceed as in Step 2 to obtain for each v_b,

$$\Delta_b(v_b, \bar{\tau}) = \frac{1}{v_b - \bar{\tau}} \left[\lim_{\epsilon \to 0} \int_{\tau-\epsilon}^{v_b} \delta(\bar{\tau} - \epsilon, \tau) d\tau - \int_{\tau+\epsilon}^{v_b} \delta(\bar{\tau} + \epsilon, \tau) d\tau \right].$$

Next,

$$\lim_{\epsilon \to 0} \int_{\tau-\epsilon}^{v_b} \delta(\bar{\tau}-\epsilon, \tau) d\tau - \int_{\tau+\epsilon}^{v_b} \delta(\bar{\tau}+\epsilon, \tau) d\tau = \lim_{\epsilon \to 0} \int_{\tau+\epsilon}^{v_b} \delta(\bar{\tau}-\epsilon, \tau) - \delta(\bar{\tau}+\epsilon, \tau) d\tau + \int_{\tau-\epsilon}^{\tau+\epsilon} \delta(\bar{\tau}-\epsilon, \tau) d\tau$$

$$= \lim_{\epsilon \to 0} \int_{\tau+\epsilon}^{v_b} \delta(\bar{\tau} - \epsilon, \tau) - \delta(\bar{\tau} + \epsilon, \tau) d\tau = \int_{(\tau, v_b]} \lim_{\epsilon \to 0} \delta(\bar{\tau} - \epsilon, \tau) - \delta(\bar{\tau} + \epsilon, \tau) d\tau,$$

where the second equality follows by MCT, and the third one by the bounded convergence theorem. Thus, for every v_b,

$$\Delta_b(v_b, \bar{\tau}) = \frac{1}{v_b - \bar{\tau}} \int_{(\tau, v_b]} \lim_{\epsilon \to 0} \delta(\bar{\tau} - \epsilon, \tau) - \delta(\bar{\tau} + \epsilon, \tau).$$

For each $k = 1, ..., \infty$, by continuity and monotonicity of $\delta(\bar{\tau} + \frac{1}{k}, \tau)$, and since $\delta(\bar{\tau} + \frac{1}{k}, \bar{\tau} + \frac{1}{k}) = 0$, there exists a $v_b^{(k)} > \bar{\tau} + \frac{1}{k}$, such that $\delta(\bar{\tau} + \frac{1}{k}, v_b^{(k)}) < \frac{1}{k}$. On the other hand, $\delta(\bar{\tau} - \frac{1}{k}, v_b^{(k)}) \geq \Delta$, so that

$$\Delta_b(v_b^{(k)}, \bar{\tau}) > \Delta - \frac{1}{k},$$

which by Step 2.1. implies that $\Delta_b(v_b, \bar{\tau}) \geq \Delta$. By a symmetric argument, it must be that $\Delta \geq \Delta_b(v_2, \bar{\tau})$. This concludes the proof of Step 2.2.

Now we can wrap up the proof of the Theorem. By Step 2.1., the set where $\delta(0, \tau)$ is continuous is a countable union of open intervals, denote this by $V = \cup_{l=1}^{\infty} (v^{(l-1)}, v^{(l)})$, where $\lim_{l \to \infty} v^{(l)} = 1$. By Step 2.2, V is also the set where $\delta(\tau, 1)$ is continuous. By Step 2.1., $\delta(v_s, v_b)$ is continuous for $v_s, v_b \in (v^{(l-1)}, v^{(l)})$, so that on $(v^{(l-1)}, v^{(l)})$ the Lemma holds by Case 1. By Steps 2.1. and 2.2. we can sum separately over jumps and over points of continuity, which concludes the proof of the theorem. □

Appendix B.

Proposition 14. Weak Type Monotonicity: In every BE, $\frac{\partial p_i(v_i, t)}{\partial v_i} \geq 0$, for all times $t \in (0, \infty)$ and types $v_i \in [0, 1]$, which satisfy the condition that $H_j(v_i, t)$ is strictly increasing at t.

Proof. Fix the buyer strategy at some $p_b(., .)$. Denote by $H_b(v_s, t; p_s)$ the mass of buyer's types with whom v_s enters in agreement until time t if she plays the strategy $p_s(., .)$. Observe that at any t, s.t. $\exists v_b$ with $p_s(v_s, t) = p_b(v_b, t)$, $H_b(v_s, t; p_s)$ is strictly increasing if and only if $p_s(v_s, .)$ is strictly decreasing or $p_b(v_b, .)$ is strictly increasing in t. This follows from continuity of $p_s(., .)$ and $p_b(., .)$ w.r.t. v. Moreover, $H_b(v_s, t; p_s)$ has a jump at t if and only if $\exists v_b', v_b''$ s.t. $p_s(v_s, t) = p_b(v_b, t)$ for all $v_b \in (v_b'', v_b')$.

We have to show that $p_s(v_s, t) \geq p_s(v_s', t)$ for any $v_s \geq v_s'$ and any t s.t. $H_b(v_s, t; p_s)$ is strictly increasing at t (at any v_s, where the condition in the statement of the lemma is satisfied, $H_b(v_s, t; p_s)$ is strictly increasing, and it can have a jump).

We proceed by contradiction. Assume there are $v_s > \hat{v}_s$ and $\hat{t}$ s.t. $p_s(v_s, \hat{t}) < p_s(\hat{v}_s, \hat{t})$ and $H_b(v_s, \hat{t}; p_s)$ is strictly increasing at $\hat{t}$. Denote by

$$t_0 = \inf \left\{ t | H_b(v_s, t; p_s) > 0, t < \hat{t}, \text{ and } p_s(v_s, t) < p_s(\hat{v}_s, t) \text{ for all } \tau \in (t, t') \right\},$$

$$t_1 = \min \left\{ t | t > \hat{t}, p_s(v_s, t) = p_s(\hat{v}_s, t) \right\}.$$

In other words, t_0 is the largest time until which the demands of v_s and $\hat{v}_s$ are monotonic, and t_1 is the first time after t_0 at which these demands are equal. First, by continuity of $p_s(v_s, .)$ and $p_s(\hat{v}_s, .)$ it is clear that $t_0 < \hat{t} < t_1$. Moreover, $t_1 < \infty$ since by the previous lemma, $\lim_{t\to\infty} p_s(v_s, t) = v_s > \hat{v}_s = \lim_{t\to\infty} p_s(\hat{v}_s, t)$, hence, by continuity there exists a $\bar{t} < \infty$ s.t. $p_s(v_s, t) > p_s(\hat{v}_s, t)$ for all $t \geq \bar{t}$. Since $H_b(v_s, \hat{t}; p_s)$ is strictly increasing at $\hat{t}$, it is also clear that $H_b(v_s, t_0; p_s) < H_b(v_s, t_1; p_s)$.

If some seller type bids lower at time t she will have agreed with a larger mass of the buyer's types. In other words, $p_s(v_s, t) \leq p_s(\hat{v}_s, t) \Rightarrow H_b(v_s, t; p_s) \geq H_b(\hat{v}_s, t; p_s)$ for all t and all v_s and $\hat{v}_s$, which follows from the monotonicity of $p_s(., .)$ and $p_b(., .)$ w.r.t. t. Applying this twice at t_0 and t_1, we get $H_b(v_s, t_0; p_s) = H_b(\hat{v}_s, t_0; p_s)$ and that $H_b(v_s, t_1; p_s) = H_b(\hat{v}_s, t_1; p_s)$. By construction, we have $p_s(v_s, t) < p_s(\hat{v}_s, t)$ for all $t \in (t_0, t_1)$. This implies that $H_b(v_s, t; p_s) \geq H_b(\hat{v}_s, t; p_s)$ for all $t \in (t_0, t_1)$.

In equilibrium, $p_s(v_s, .)$ is the optimal strategy for type v_s, and $p_s(\hat{v}_s, .)$ is optimal for type $\hat{v}_s$ on the interval (t_0, t_1). In particular (from now on we omit subindexes and write $p_s(v_s, t) = p(t)$, $p_s(\hat{v}_s, t) = \hat{p}(t)$, $H_b(v_s, t; p_s) = H(t)$ and $H_b(\hat{v}_s, t; p_s) = \hat{H}(t)$)

$$\int_{t_0}^{t_1} e^{-t} (p(t) - v_s) \, dH(t) \geq \int_{t_0}^{t_1} e^{-t} (\hat{p}(t) - v_s) \, dH(t) \tag{11}$$

and

$$\int_{t_0}^{t_1} e^{-t} (\hat{p}(t) - \hat{v}_s) \, d\hat{H}(t) \geq \int_{t_0}^{t_1} e^{-t} (p(t) - \hat{v}_s) \, d\hat{H}(t). \tag{12}$$

Subtracting these two inequalities, we obtain

$$\int_{t_0}^{t_1} e^{-t} dH(t) \leq \int_{t_0}^{t_1} e^{-t} d\hat{H}(t).$$

Integrate by parts to get $\int_{t_0}^{t_1} e^{-t} dH(t) = H(t_1) e^{-t_1} - H(t_0) e^{-t_0} + \int_{t_0}^{t_1} e^{-t} H(t) \, dt$, and similarly for the right hand side. Now we use $H(t_0) = \hat{H}(t_0)$ and $H(t_1) = \hat{H}(t_1)$, to obtain

$$\int_{t_0}^{t_1} e^{-t} H(t) \, dt \leq \int_{t_0}^{t_1} e_b^{-t} \hat{H}(t) \, dt.$$

But since $H(t) \geq \hat{H}(t)$ for all $t \in (t_0, t_1)$ the last inequality implies that it must in fact be $H(t) = \hat{H}(t)$ for almost all $t \in (t_0, t_1)$. Now take for example (11), and rewrite it into

$$\int_{t_0}^{t_1} e^{-t} \left(p\left(t\right) - \hat{p}\left(t\right)\right) dH\left(t\right) \geq 0.$$

But $\hat{p}(t) > p(t)$ for $t \in (t_0, t_1)$, which implies that

$$\int_{t_0}^{t_1} e^{-t} \left(p\left(t\right) - \hat{p}\left(t\right)\right) dH\left(t\right) < 0,$$

which is a contradiction. □

Lemma 15. Total concession at infinity: In a regular PBE, $\lim_{t\to\infty} p_i(v_i, t) = v_i$ for all $v_i \in [0,1]$.

Proof. Denote $P_i(v_i) = \lim_{t\to\infty} p_i(v_i, t)$. The proof is divided into three steps. In Step 1 we show that $P_s(1) = 1$ (which holds trivially) and the continuity at 1 imply that $P_s(0) = 0$. In Step 2 we show that $P_s(.)$ is a continuous function, hence it attains all values in the interval $[0,1]$. Finally, in Step 3 we show that the claim is true for the seller. An analogous proof works for the buyer.

Step 1: $P_s(0) = 0$. Suppose this isn't the case, i.e. $P_s(0) = K > 0$ in equilibrium. Denote by $p_s(0,t)$ such equilibrium strategy for the seller, and by $p_b(v_b, t)$ the equilibrium strategy of the buyer of type v_b. By individual rationality we have that $P_b(0) = 0$. Also by individual rationality, we have that $P_b(v_b)$ is bounded above, i.e. $P_b(v_b) \leq v_b$. Since $P_b(v_b) \geq 0$, these imply that $P_b(v_b)$ is continuous at point $v_b = 0$. From continuity of P_b around $v_b = 0$ we get that there is a positive mass of types $v_b \in [0,1]$ for which $P_b(v_b) < K$. But then the seller of type 0 could improve her expected payoff by playing p_s until some large time t', and then lowering her demand to 0, according to some strategy p'_s. To see this, notice that p_s and p_b are continuous and for all v_b, $p_s(0,t)$ is non-increasing and $p_b(v_b, t)$ is non-decreasing in t. Thus the support of $g_b(v_b|t)$ is shrinking as time elapses. When t is very large, the support of $g_b(v_b|t)$ will be very close to the ex-post belief when no agreement has been reached. Hence t' is given as the moment when the expected continuation payoff of playing p_s, conditional on $v_b \geq K$, is lower than the expected continuation payoff of playing p'_s, conditional on $v_b > 0$. This establishes the contradiction. The same argument shows that $P_s(v_s)$ is continuous in a neighbourhood of the point $v_s = 0$.

Step 2. Assume thus that $P_s(v_s)$ is discontinuous at $\bar{v}_s$, i.e. $P_s(\bar{v}_s) = \hat{l}$ and $\lim_{v_s \searrow \bar{v}_s} = \bar{l}$, where $\bar{l} > \hat{l}$. Then there must exist an $\bar{v}_b$ s.t. $P_b(\bar{v}_b) = \bar{l}$, and $\lim_{v_b \nearrow \bar{v}_b} P_b(v_b) = \hat{l}$ (same argument as in Step 1, left-continuity of P_s and right-continuity of P_b). Take a $\hat{v}_s > \bar{v}_s$. By continuity of p_s in t, there exists an M_s s.t. $p_s(\bar{v}_s, t) - \hat{l} < \varepsilon$ for all $t \geq M_s$. Also, notice that $p_s(\hat{v}_s, t) \geq \bar{l}$. Now fix $\varepsilon = \frac{\bar{l}-\hat{l}}{4} > 0$ and take a $t \geq M_s$. Then at t, $p_s(\bar{v}_s, t) < \hat{l} + \varepsilon$ while $p_s(\hat{v}_s, t) \geq \bar{l}$ for all $\hat{v}_s > \bar{v}_s$, contradicting the continuity of p_s in v_s. This proves that $P_s(v_s)$ has to be right-continuous. By assumption, $P_s(v_s)$ is left-continuous, [9] hence it is continuous. In step 1 we proved that $P_s(1) = 1$ and $P_s(0) = 0$, so by Rolle's theorem it attains all values between 0 and 1.

Step 3: $P_s(v_s) = v_s$ for all $v_s \in [0,1]$. Take an $v_s \in (0,1)$. By steps 1 and 2, P_s takes all the values in the interval $[0,1]$ and is continuous (thus measurable), strictly positive on $(0,1]$. Thus we can define the measure μ_s

$$\mu_s(V) = \int_V P_s(v)\, dm(v) \text{ for each mesaurable } V \subset [0,1],$$

where $m(.)$ denotes the usual Lebesgue measure. By strict positivity, continuity, and boundedness of $P_s(v_s)$, μ_s is an equivalent measure to m. Now suppose that $P_s(v_s) > v_s$. By equivalence of μ_s to m there exists a positive mass of types v_b s.t. $p_b(v_b) \in (v_s, P_s(v_s))$. To see this define $B = \{v_b | p_b(v_b) \in (v_s, P_s(v_s))\}$. Since μ_s and m are equivalent, $m(B) > 0$. Now repeat the same argument as in Step 1 to get a contradiction. Hence indeed $P_s(v_s) = v_s$.

□

Recall that the entry time $t_i^E(v_i)$ is the first time when v_i could agree with some type of player j, $t_i^E(v_i) = \min\{t \mid \tilde{v}_j(v_i, t) \neq \emptyset\}$. It is easy to see that at $t_i^E(v_i)$ the demand of type v_i must be compatible exactly with that of the weakest type of the opponent.

Lemma 16. Initial proposal and entry time: In an undominated regular BE $p_s(v_s, t_s^E(v_s)) = p_b(1, t_s^E(v_s))$ and $p_b(v_b, t_b^E(v_b)) = p_s(0, t_b^E(v_b))$, for all $v_i \in [0,1]$.

Proof. Denote by $\gamma_i(v_i)$ the starting point of the bids of type v_i: $\gamma_i(v_i) = \lim_{t \searrow 0} p_i(v_i, t)$. We will prove that $\gamma_s(0) = \gamma_b(1)$, which immediately proves the Lemma. In an equilibrium the type $v_s = 0$ at time 0 demands a share that will give her a positive probability of agreement in at least a very short time - otherwise each type of every agent would know that there was some dead delay at the start where the only thing that would happen would be that agents would lower their demands up to the point where the minimal-cost-seller and the maximal-valuation-buyer could agree, violating that the BE is undominated. On the other hand, it cannot be that she bids a price which meets the bid of some buyer of type $v_b^0 < 1$ - meaning that $\gamma_s(0) = \gamma_b(v_b^0)$. This follows from the price sharing rule in case that bids are more than compatible, since then the type $v_s = 0$ could profitably deviate by starting with a bid that meets

[9] Type $\bar{v}_s$ is at $t = \infty$ indifferent between demanding $\hat{l}$ and $\bar{l}$; the former doesn't improve her probability of reaching an agreement since the mass of opposing types with bids between $\bar{l}$ and $\hat{l}$ is 0. However, by an argument similar to the proof of Step 1, we can argue, that she doesn't lose anything by bidding $\hat{l}$, which gives us left-continuity of P_s. Left-continuity of P_s is thus essentially an assumption on how agents resolve their indifference at the horizon.

type $v_b = 1$. Then she would "rip off" all the excess agreement profits by lowering her bid very rapidly to $\gamma_b(v_b^0)$. By making her move fast enough it is clear that such deviation could be profitable. □

Thus for all sellers except the minimal-cost-type it is in equilibrium optimal to wait with a high bid for a while. Symmetrically for the buyers. It means that there will necessarily be delays with probability 1.

We remark that in each undominated regular BE $t_i^E(v_i) < \infty$ if and only if $v_s < 1, v_b > 0$. Otherwise the strategy of v_i would be strictly dominated.

We now write down the dynamic optimization problem. In equilibrium, agents maximize payoffs, given the type-contingent strategies of the other player. Thus, agents are picking optimal functions $p_i(v_i, \cdot)$, $i = s, b$, determining how bids change over time.

An important step in the proof of proposition 17 below is to show that for every $(v_i, t) \in [0,1] \times [t_i^E(v_i), \infty)$,$\tilde{v}_j(v_i, t)$ is a function (and not a correspondence), defined by

$$p_j(\tilde{v}_j(v_i, t), t) = p_i(v_i, t). \tag{13}$$

This is a consequence of the assumption that the opponent plays a strictly type-monotone strategy, and the implicit function theorem.

Proposition 17. Optimization Program: If the strategy of agent j is regular and separating, then the best reply of agent i of type v_i solves the following optimization program

$$Max_{p_i(v_i,\cdot)\in\Pi_i} \int_{[t_i^E(v_i),\infty)} e^{-t} u_i(p_i(v_i,t), s_i) g_j(\tilde{v}_j(v_i,t)) \frac{\partial \tilde{v}_j(v_i,t)}{\partial t} dt,$$

$$\text{s.t. (13) and } t_i^E(v_i) \text{ defined by } \tilde{v}_b\left(v_s, t_s^E(v_s)\right) = 1 \text{ or } \tilde{v}_s\left(v_b, t_b^E(v_b)\right) = 0.$$

Proof. Consider the seller and fix her type to be v_s. When entering into negotiations at $t_s^E(v_s)$, she decides her optimal concession plan $p_s(v_s, t)$, $t > t_s^E(v_s)$, in order to maximize her expected discounted future payoff. Denote by $H_b(t)$ the probability of seller v_s reaching agreement up to time t (we omit the parameter v_s in $H_b(t; v_s)$). The seller is solving the following program

$$Max_{p_s(v_s,\cdot)\in\Pi_s} \int_{[t_s^E(v_s),\infty)} e^{-t}(p_s(v_s,t) - v_s) dH_b(t)$$

But the possibility of reaching an agreement at some $t > t_s^E(v_s)$ is exactly the possibility that the seller's bid will at t meet the bid of some type of the buyer.

For each $t \geq t_s^E(v_s)$, recall that $\tilde{v}_b(v_s, t)$ is the type of buyer with whom v_s reaches agreement at moment t. Thus $\tilde{v}_b(v_s, t)$ is implicitly defined from the relation

$$p_b(\tilde{v}_b(v_s, t), t) = p_s(v_s, t).$$

Type monotonicity implies that at every instant there will be at most one type reaching agreement with each type of the other agent. Thus, by definition of t_i^E $\tilde{v}_b\left(v_s, t_s^E(v_s)\right) = 1$, and by Lemma 15 $\lim_{t\to\infty} \tilde{v}_b(v_s, t) = v_s$. Taking the derivative with respect to t, we can express

$$\frac{\partial \tilde{v}_b(v_s, t)}{\partial t} = \frac{\frac{\partial p_s(v_s,t)}{\partial t}}{\frac{\partial p_b(\tilde{v}_b(v_s,t),t)}{\partial v_b}}.$$

By assumption, $\frac{\partial p_i}{\partial t}$ are both finite, $\frac{\partial p_s}{\partial t} \leq 0$ and $\frac{\partial p_b}{\partial t} \geq 0$. Hence type monotonicity, and the implicit function theorem imply that, that for each $t \geq t_i^E(s_i)$, $\tilde{v}_b(v_s, t)$ is a well defined differentiable function of time, with $0 \leq \left|\frac{\partial \tilde{v}_b(v_s,t)}{\partial t}\right| < \infty$. In other words, at every $t \geq t_s^E(v_s)$ there exists exactly one type $\tilde{v}_b(v_s, t)$ of the buyer, with whom v_s would reach agreement at that moment. These facts have two consequences. First, the probability of reaching an agreement by t, $H_b(t)$, has no mass points because the distribution of types of the buyer has no mass points. Second, the marginal increase in $H_b(t)$, $dH_b(t)$, is equal to the marginal increase of the mass of buyer's types that the seller would agree with by moment t. Also, the seller knows that before $t_s^E(v_s)$ her bids were unrealistic, so she cannot update her beliefs until that moment. Since $\tilde{v}_b$ is differentiable with respect to time, the beliefs are updated continuously and differentiably from $t_s^E(v_s)$ on. In other words, we have established that at $t_s^E(v_s)$ the belief of the seller is exactly $G_b(v_b)$, and at every moment $dH_b(t) = dG_b(\tilde{v}_b(v_s, t)) = g_b(\tilde{v}_b(v_s, t)) \frac{\partial \tilde{v}_b(v_s,t)}{\partial t} dt$. This completes the proof for the seller. The case of the buyer is analogous. □

The optimization problem stated in Proposition 17 can be best approached as a problem where i is choosing two unknown functions $p_i(v_i, \cdot)$ and $\tilde{v}_j(v_i, .)$ which are bound by the constraint (13), where $p_j(\cdot, \cdot)$ is a given and fixed function (the strategies of all possible types of agent j). Good references for the calculus of variations are Elsgolts [1970] and Troutman[1995].

The optimality condition at the lower boundary of optimization is given by definition of $t_i^E(v_i)$ - implicitly written as $\tilde{v}_b(v_s, t_s^E(v_s)) = 1$ or $\tilde{v}_s(v_b, t_b^E(v_b)) = 0$. In the following lemma we provide the first order condition for the optimization program of agent i, for $t > t_i^E(v_i)$. To save on cumbersome notation we omit several unambiguous arguments in the functions.

Lemma 18. First order condition: In a regular and separating BE, strategies $p_i(v_i, .)$, $i = s, b$, satisfy the following first order conditions

$$\begin{array}{l} (p_s - v_s) = \left(\frac{\partial p_b(\tilde{v}_b,t)}{\partial \tilde{v}_b} \frac{d\tilde{v}_b}{dt} - \frac{\partial p_s}{\partial t} \right), \forall v_s \in [0,1], \forall t > t_s^E(v_s); \\ (v_b - p_b) = -\left(\frac{\partial p_s(\tilde{v}_s,t)}{\partial \tilde{v}_s} \frac{d\tilde{v}_s}{dt} - \frac{\partial p_b}{\partial t} \right), \forall v_b \in [0,1], \forall t > t_b^E(v_b). \end{array} \tag{14}$$

Proof. We fix v_i and economize the notation to write $\tilde{v}_j(v_i, t) = \tilde{v}_j$ and $\frac{\partial \tilde{v}_j(v_i,t)}{\partial t} = \dot{\tilde{v}}_j$. We write the Hamiltonian

$$\begin{aligned} H_i(t) &= e^{-t} u_i(p_i(v_i, t), v_i) g_j(\tilde{v}_j) \dot{\tilde{v}}_j - \\ &\quad -\mu(t) \left(p_j(\tilde{v}_j, t) - p_i(v_i, t) \right), \end{aligned}$$

and compute the Euler conditions for the unknown functions

$$\begin{aligned} \frac{\partial H_i}{\partial \tilde{v}_j} &= e^{-t} u_i(p_i(v_i, t), v_i) g_j'(\tilde{v}_j) \dot{\tilde{v}}_j - \mu \frac{\partial p_j(\tilde{v}_j, t)}{\partial \tilde{v}_j}, \\ \frac{d}{dt} \frac{\partial H_i}{\partial \dot{\tilde{v}}_j} &= e^{-t} u_i(p_i(v_i, t), v_i) g_j'(\tilde{v}_j) \dot{\tilde{v}}_j + \\ &\quad + e^{-t} \frac{\partial u_i(p_i(v_i, t), v_i)}{\partial p} \frac{\partial p_i(v_i, t)}{\partial t} g_j(\tilde{v}_j) + \\ &\quad - e^{-t} u_i(p_i(v_i, t), v_i) g_j(\tilde{v}_j), \\ \frac{\partial H_s}{\partial p_i} &= e^{-t} \frac{\partial u_i(p_i(v_i, t), v_i)}{\partial p} g_j(\tilde{v}_j) \dot{\tilde{v}}_j + \mu, \\ \frac{\partial H_s}{\partial \dot{p}_i} &= 0. \end{aligned}$$

Whence we have the two Euler equations

$$\begin{array}{l} -\mu \frac{\partial p_j(\tilde{v}_j,t)}{\partial \tilde{v}_j} - e^{-t} \frac{\partial u_i(p_i(v_i,t),v_i)}{\partial p} \frac{\partial p_i(v_i,t)}{\partial t} g_j(\tilde{v}_j) + e^{-t} u_i(p_i(v_i, t), v_i) g_j(\tilde{v}_j) = 0, \\ e^{-t} \frac{\partial u_i(p_i(v_i,t),v_i)}{\partial p} g_j(\tilde{v}_j) \dot{\tilde{v}}_j + \mu = 0. \end{array}$$

From the second Euler equation we can eliminate μ and the density g_j also disappears from the first to obtain the condition

$$\begin{aligned} u_i(p_i, v_i) &= \frac{\partial u_i(p_i, v_i)}{\partial p_i} \left(\frac{\partial p_j(\tilde{v}_j, t)}{\partial \tilde{v}_j} \frac{d\tilde{v}_j}{dt} - \frac{\partial p_i}{\partial t} \right), \\ \text{for } t &\geq t_E(v_i), i = s, b. \end{aligned}$$

□

Lemma 18 has two important implications. The first is that a best response to a strictly type-monotone strategy is strictly type-monotone.

Corollary 19. If $p_i(.,.)$ is a best response to a regular strategy $p_j(.,.)$, such that $\frac{\partial p_j}{\partial v_j} > 0$, then $\frac{\partial p_i}{\partial v_i} > 0$.

Proof. Let $i = s$ and assume that $\frac{\partial \tilde{v}_b}{\partial t} = 0$. By assumption, $\frac{\partial p_b(\tilde{v}_b,t)}{\partial v_b} > 0$ and $\frac{\partial p_s}{\partial t} \leq 0$, so that $\frac{\partial \tilde{v}_b}{\partial t} = 0$ contradicts equation (14). □

The second implication of Lemma 18 is that the equilibrium strategies must be independent of players' beliefs. Hence the following is immediate (see for instance Ledyard [1978]).

Corollary 20. A PBE in regular and strictly type monotone strategies must be an ex-post equilibrium.

Combining Corollary 20 with Lemma 15 yields Theorem 12.

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